Lecture February, 25, 2013

1. Let $X_1, X_2, \cdots, X_n$ i.i.d random variables distributed as $\gamma(a, r)$. Show that the average,
$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n},$$
is distributed as $\gamma(na, nr)$. Compare with the distribution of the sum.

2. The duration in **thousangds of hours** of a certain type of light bulbs is exponentially distributed $\exp(\lambda)$, with mean 2. A system has been installed with 10 light bulbs connected in such a way that when one fails, the next one takes over.

   (a) What is the probability that a single light bulb lasts more than 6000 hours?

   (b) What is the distribution of the total time the system provides light (i.e., the time until the last of the 10 light bulbs connected fails)?

   (c) What is the probability that such lightning system lasts no more than 23800 hours?

   (d) If we had 100 rather than 10 light bulbs, what would be the mean and variance of the total time until the last fails? What would be the probability that this time exceeds 210000 hours?

3. Let X_1, X_2, X_3, X_4 and X_5 be independent random variables with distributions $N(3, \sigma^2 = 4)$ (X_1, X_2 and X_3) and $N(5, \sigma^2 = 4)$ (X_4 and X_5).

   (a) Defining,
$$Y = \frac{2\left(\frac{X_1-3}{2}\right)}{\sqrt{\left(\frac{X_2-3}{2}\right)^2 + \left(\frac{X_3-3}{2}\right)^2 + \left(\frac{X_4-5}{2}\right)^2 + \left(\frac{X_5-5}{2}\right)^2}},$$
   compute:

   i. $P(Y \leq 1.53)$

   ii. $P(Y \in (-0.741, 2.13))$

   iii. The value k such that $P(Y > k) = 0.2$

   iv. The value k such that $P(Y \leq k) = 0.3$

   (b) Defining
$$Z = \frac{3\left[(X_4 - 5)^2 + (X_5 - 5)^2\right]}{2\left[(X_1 - 3)^2 + (X_2 - 3)^2 + (X_3 - 3)^2\right]},$$
   compute:

i. The value k such that $P(Z \leq k) = 0.9$
ii. The value k such that $P(Z > k) = 0.95$
iii. $P(Z > 9.55)$
iv. $P(Z \in (0.109, 5.46))$

(c) If we define

$$V = \left(\frac{X_1 - 3}{2}\right)^2 + \left(\frac{X_2 - 3}{2}\right)^2 + \left(\frac{X_3 - 3}{2}\right)^2 + \left(\frac{X_4 - 5}{2}\right)^2 + \left(\frac{X_5 - 5}{2}\right),$$

compute:

i. The value k such that $P(V \leq k) = 0.9$
ii. The value k such that $P(V > k) = 0.95$
iii. $P(V > 1.61)$
iv. $P(V \in (2.67, 12.8))$

4. Let X_1, X_2, X_3, X_4 and X_5 mutually independent random variables, with X_1, X_2 and X_3 distributed as $N(2, \sigma^2 = 4)$ and X_4, X_5 distributed ad $N(1, \sigma^2 = 1)$. Obtain from them random variables with the following distributions: χ^2_4, $\mathcal{F}_{1,4}$, t_4.