Lecture March, 18, 2013

1. Let X be distributed with density function $f_X(x;\theta) = \frac{1}{\theta}e^{x/\theta}$, for $x > 0$. (REMARK: This is the usual exponential distribution parameterized in terms of the mean θ; an alternative parameterization is in terms of $\lambda = 1/\theta$, in which case the mean is $1/\lambda$ and the variance $1/\lambda^2$)
   (a) Obtain the MLE of θ based on a sample of n independent observations.
   (b) Is it: i) Consistent? ii) Unbiased?
   (c) Is the MLE estimator of λ unbiased?
2. Consider the MLE estimator of θ in a $U(0, \theta)$ distribution, based on n independent observations $X_1, \ldots, X_n$. We already know know $\hat{\theta}_{MLE} = X_{(n)} = \max\{X_1, \ldots, X_n\}$.
   (a) Show that it is consistent. (Hint: You can follow a direct approach here. Fix a neighborhood of θ of arbitrary width ϵ and show that for sufficiently large n, the probability of being there is as close to 1 as desired.)
   (b) Find the distribution, then the density of $\hat{\theta}_{MLE} = X_{(n)}$. (HINT: $F_{\hat{\theta}_{MLE}}(x) = P(\hat{\theta}_{MLE} \leq x) = P(\cap_{i=1}^{n} X_i \leq x)$.)
   (c) Find the mean of $\hat{\theta}_{MLE}$. Remove the bias to obtain an unbiased estimator $\hat{\theta}_U$.
   (d) Find the variance of $\hat{\theta}_U$. (HINT: Make use of the fact that $\sigma^2 = \alpha_2 - m^2$; you have computed the mean m in the previous step, so you only need α_2.)
   (e) What is the convergence rate to zero of the variance you have found?
3. When we take a sample of n independent observations $X_1, \ldots, X_n$ from a $b(p)$ distribution, we have found that the MLE is $\overline{X}$, the number of "ones" divided by the number of throws. Show that it is is consistent.