P

Statistics Applied to Economics

Degree in Economics

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Índice I

Probability function

- Defined on non-negative integers, $x = 0, 1, 2, \dots$ with $P_X(x)$:

$$P_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

- Well defined; obviously non-negative, and:

$$\begin{aligned} \sum_{x=0}^{\infty} P_X(x) &= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} \\ &= e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots\right) \\ &= e^{-\lambda}e^{\lambda} = 1 \end{aligned}$$

How do we get last expression from the previous one?

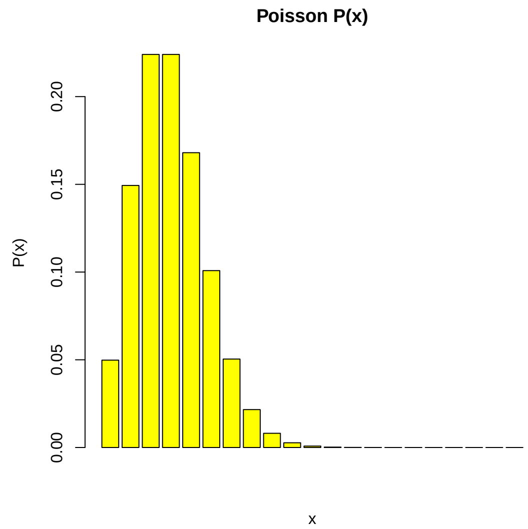
Using the Taylor series expansion $e^t = 1 + t + t^2/2! + t^3/3! + \dots$

Historical notes

- Named after Siméon Denis Poisson (1781-1840)
- French mathematician, contemporaneous of Lagrange, Laplace and Fourier.
- Did important work in many areas of mathematics.
- See `http://en.wikipedia.org/wiki/Siméon_Denis_Poisson`.

What does it look like?

```
> x <- 0:1
> dpois(x,lambda=3)
[1] 0.04978707 0.14936121
> x <- 0:20
> barplot(dpois(x,lambda=3),
       col="yellow",
       xlab="x",
       ylab="P(x)",
       main="Poisson P(x)")
```



Moment generating function

$$\begin{aligned}
\varphi_X(u) \stackrel{\text{def}}{=} E[e^{uX}] &= \sum_{x=0}^{\infty} e^{uX} P_X(x) = \sum_{x=0}^{\infty} e^{uX} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^u)^x}{x!} \\
&= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^u)^x}{x!} \qquad (1) \\
&= e^{-\lambda} e^{\lambda e^u} \qquad (2) \\
&= e^{\lambda(e^u - 1)}
\end{aligned}$$

How do we get (2) from(1) in previous line?

Yet another use of $e^t = 1 + t + t^2/2! + t^3/3! + \ldots$

Mean and variance

- Remember:

$$\alpha_1 = \left[\frac{\partial \varphi_X(u)}{\partial u}\right]_{u=0} \qquad \alpha_2 = \left[\frac{\partial^2 \varphi_X(u)}{\partial u^2}\right]_{u=0}$$

- Hence,

$$\begin{aligned}
\alpha_1 &= \left[\frac{\partial}{\partial u} e^{\lambda(e^u-1)}\right]_{u=0} \\
&= \left[\frac{\partial\left(\lambda(e^u - 1)\right)}{\partial u} \times e^{\lambda(e^u-1)}\right]_{u=0} \\
&= \left[\lambda e^u e^{\lambda(e^u-1)}\right]_{u=0} = \lambda \\
\alpha_2 &= \left[\frac{\partial^2}{\partial u^2} e^{\lambda(e^u-1)}\right]_{u=0} = \lambda + \lambda^2
\end{aligned}$$

How are $\alpha_1 = E[X]$ and $\alpha_2 = E[X^2]$ related to meand and variance?

$m = \alpha_1 = \lambda$ and $\sigma^2 = \alpha_2 - (\alpha_1)^2 = \lambda$.

Sum of independent Poisson variables

- Let $X_i \sim \mathcal{P}(\lambda_i)$ for $i = 1, \ldots, n$, independent of each other.
- Let $X = X_1 + \ldots + X_n$. Then, $X \sim \mathcal{P}(\lambda_1 + \ldots + \lambda_n)$.
- Proof is easy:

$$\begin{aligned} \varphi_X(u) &= \varphi_{X_1}(u) \times \cdots \times \varphi_{X_n}(u) \\ &= e^{\lambda_1(e^u-1)} \times \cdots \times e^{\lambda_n(e^u-1)} \\ &= e^{(\lambda_1+\ldots+\lambda_n)(e^u-1)} \end{aligned}$$

and we recognize in the last expression the mgf of a Poisson random variable with $\lambda = \lambda_1 + \ldots + \lambda_n$.

Would *the average* of $X_1, \ldots, X_n$ be Poisson-distributed?

No, $\varphi_{\overline{X}}(u) = e^{(\lambda_1+\ldots+\lambda_n)(e^{u/n}-1)}$ which is not the mgf of a Poisson.

Poisson as a limit of the binomial

- Remember: if we have a sequence of random variables Z_n and

$$\lim_{n\to\infty} \varphi_{Z_n}(u) \to \varphi_Z(u)$$

then the distribution of Z_n approaches the distribution of Z

- Now, consider $Z_n \sim b(p = \lambda/n, n)$, We have,

$$\begin{aligned} \varphi_{Z_n}(u) &= [q + pe^u]^n = [(1-p) + pe^u]^n \\ &= [1 + p(e^u - 1)]^n \\ &= \left[1 + \frac{\lambda}{n}(e^u - 1)\right]^n \\ \lim_{n\to\infty} \varphi_{Z_n}(u) &= \lim_{n\to\infty} \left[1 + \frac{\lambda(e^u-1)}{n}\right]^n = e^{\lambda(e^u-1)} \end{aligned}$$

which is $\varphi_Z(u)$ of a Poisson distribution with parameter λ.

Remember what additional condition was required on $\varphi_Z(u)$?

It has to be continuous for $u = 0$.

Practical use of the limiting distribution (I)

- Whenever $np \to \infty$, normal approximation better.
- Poisson approximation best for $\lambda = np < 18$.
- Particularly useful when np very small (in which case normal approximation is quite poor).
- Discrete approximation with a discrete distribution: no continuity corrections, no nothing.
- Poisson $P_X(x) = e^{-\lambda}\lambda^x/x!$ quite easy to compute, even on a pocket calculator.

What problems would you anticipate calculating $P_X(x)$?

Large factorials might be the only problem ($69! = 1.711225 \times 10^{98}$).

Practical use of the limiting distribution (II)

- Tables do exist.
- We have the usual assortment of `{d,p,q,r}pois` functions in R, to assist with any computations.
- A useful recurrence:

$$P_X(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} = \underbrace{\frac{e^{-\lambda}\lambda^{(x-1)}}{(x-1)!}}_{P_X(x-1;\lambda)} \times \frac{\lambda}{x}$$

so each probability can be obtained from the previous multiplying by $\frac{\lambda}{x}$. (First one, $P_X(0;\lambda) = e^{-\lambda}$.)

- Avoids large factorials.

Practical use of the limiting distribution (III)

```
> dbinom(x=2,size=50,prob=0.1)        # Exact binomial
[1] 0.0779429
> dpois(x=2,lambda=50*0.1)            # Poisson approximation
[1] 0.08422434
> pnorm((2.5-5)/sqrt(50*0.1*.9)) - pnorm((1.5-5)/sqrt(4.5))
[1] 0.06981634
> dbinom(x=2,size=500,prob=0.01)        # Exact binomial
[1] 0.08363103
> dpois(x=2,lambda=500*0.01)            # Poisson approximation
[1] 0.08422434
> pnorm((2.5-5)/sqrt(500*0.01*.99)) - pnorm((1.5-5)/sqrt(4.95))
[1] 0.07273327
```

The "rare events" model

- Many units, n, with small probability p of failure, and $np < 18$ give a Poisson-distributed number of units failing.
- Examples:
  - Many soldiers, small probability of dying by horse kick $\Rightarrow$ number of soldiers dead approximately Poisson-distributed.
  - Many phone lines, small probability of one of them being in use $\Rightarrow$ simultaneaous calls placed at any one moment Poisson-distributed.
  - Many houses insured against fire, small probability of any of them catching fire in the insurance period $\Rightarrow$ total number of claims in that period Poisson-distributed.
  - Arrival intervals i.i.d. exponentially distributed, $f_X(x) = \theta e^{-\theta x}$ $\Rightarrow$ total number of arrivals in $(T, T+t)$ Poisson-distributed with $\lambda = \theta t$.

Example 1 (I)

Consider a company with 120 workers. On average, they spend 10% of their time calling to the outside. They place calls independently of each other.

- What is the mean value of the number of people simultaneously calling outside?
- If there are 16 outgoing phone lines, what is the probability of no saturation?
- If the company is split in two divisions, with respectively 80 and 40 people and 10 and 6 phone lines, what's the probability of neither division having saturation?
- What are your conclusions? Is it better to provide a centralized service or not?

Example 1 (II)

- What is the mean value of the number of people simultaneously calling outside?
- If there are 16 outgoing phone lines, what is the probability of no saturation?
- Two divisions, with respectively 80 and 40 people and 10 and 6 phone lines. Probability of neither division having saturation?

```
> 120 * 0.1
[1] 12
> #
> ppois(16,lambda=12)
[1] 0.898709
> #
> ppois(10,80*0.1) *
   ppois(6,40*0.1)
[1] 0.7255885
```

Example 2

Your are auditing a company. They claim high quality of their records, with a proportion of 0.1% at most containing errors. You screen 4000 records, uncovering 6 mistakes (i.e., a proportion of 0.15%, or 50% larger than their alleged error rate). What would you conclude about the veracity of their claims?

- Assuming their claims are right, total number of errors in 4000 records Poisson distributed, with $\lambda = 4000 \times 0.001 = 4$ in the worst case.
- If $\lambda = 4$, the probability of over 5 errors is

  ```
  > 1 - ppois(5,lambda=4)
  [1] 0.2148696
  ```

  which is by no means small.
- There is no conclusive evidence to challenge their claim: with $\lambda = 4$, 6 errors out of 4000 records is by no means abnormal.

Example 3

Five hundred school children enjoy recreation. The probability that any of them injures itself and comes to the infirmary of the school to have a wound bandaged is $p = 0.01$. How many bandages must the infirmary stock at the beginning of the day so that the probability of running out is less than 0.001?

- The number of children injured is distributed as $\mathcal{P}(\lambda = 5)$.
- Bandages required are less than or equal

  ```
  > qpois(0.999,lambda=5)
  [1] 13
  ```

  with probability 0.999, so enough to stock 13.
- Let's check:

  ```
  > 1 - ppois(12:13,lambda=5)
  [1] 0.002018852 0.000697990
  ```

  We see indeed that 12 would not be enough and 13 is.

Example 4

The probability of a type of cancer in children of school age is 0.001 per children-year (=1 out of 1000 children on the average). You are suspicious of the mobile phone antennas erected in the vicinity of your district public shool, and find out that out of 400 children, 3 have contracted the disease. Is that an abnormal incidence rate?

- The number of cancer cases is distributed as $\mathcal{P}(\lambda = 0.4)$.
- The probability of less than or equal to 0, 1, 2, 3, 4 cases is:

  ```
  > ppois(0:4,lambda=0.4)
  [1] 0.6703200 0.9384481 0.9920737 0.9992237 0.9999388
  ```

  so 3 cases is fairly rare, happening by pure chance less than 1% of the time.

Example 4 (continued)

Setup like of the previous example. You collect data on all 1300 schools with 400 children each within 200m of mobile phone antennas. Have 540 cases of cancer in all, worst one alone had 4 cases. What would you say?

- Total number of cases is $\mathcal{P}(\lambda = 0.4 \times 1300)$. Then,

  ```
  > 1 - ppois(539,lambda=1300*0.4)
  [1] 0.1956853
  ```

  doesn't look abnormal; expected about 19% of the time.
- The school with 4 cases does look abnormal *in isolation:*

  ```
  > 1 - ppois(3,lambda=0.4)
  [1] 0.0007762514
  ```

- As the worst case among the 1300 schools examined, it can no longer be considered abnormal:

  ```
  > 1 - ( ppois(3,lambda=0.4) )^1300
  [1] 0.6356057
  ```

Our next week or so...

...will be a crossing of the desert.

What is ahead of us

- We need to introduce quite a few distributions.
- The fastest presentation requires that for a while we forgo applications.
- You may have a feeling of lack of purpose...
- ...but trust me:
  - I have been there, I have done that,
  - and (after some 30 years experience) think there is no better way.
- So bear with me for a while, later the subject will be as interesting as it gets.

The gamma function $\Gamma(r)$

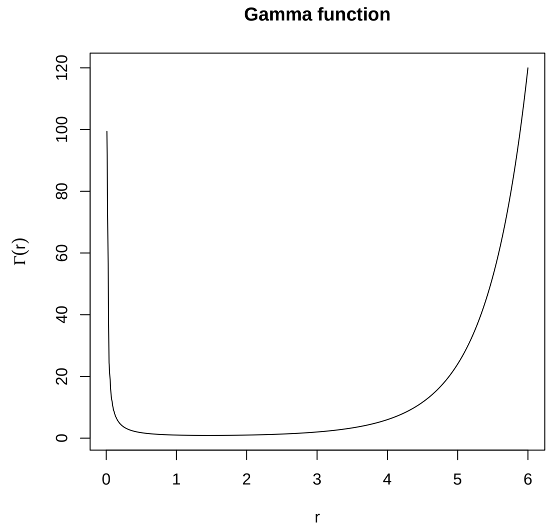
- Defined as:
$$\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$$
- Defined for all r, although only for $r > 0$ it will be of interest to us.
- Sometimes called Euler integral of the second kind.
- Does not have closed form; value can be computed analytically for certain values of r, numerically for others.
- Interestingly, $\Gamma(r) = (r-1)!$ for natural r.

How do you think $\Gamma(r)$ changes with r?

Clearly, $\Gamma(r) \to \infty$ as $r \to \infty$, but also as $r \to 0$.

$\Gamma(r)$ in R

```
> gamma(5)
[1] 24
> factorial(4)
[1] 24
> curve(gamma,from=0.01,
      to=6,n=200,
      ylab=expression(
            Gamma(r)
          ),
      xlab="r",
      main="Gamma function")
```



The gamma distribution $\gamma(a, r)$ (I).

- It is clear that

$$F_X(x) = \frac{1}{\Gamma(r)} \int_0^x t^{r-1} e^{-t} dt$$

is a well defined distribution on $[0, \infty)$.

- If we make the change $t \to at$ for $a > 0$ right hand side still defines the $\gamma(a, r)$ distribution function:

$$\frac{a^r}{\Gamma(r)} \int_0^x t^{r-1} e^{-at} dt$$

- Density function therefore is:

$$f_X(x) = \frac{a^r}{\Gamma(r)} t^{r-1} e^{-at}$$

The gamma distribution $\gamma(a, r)$ (II).

- Alternative parameterizations:

$$\begin{aligned} f_X(x) &= \frac{a^r}{\Gamma(r)} t^{r-1} e^{-at} \\ f_X(x) &= \frac{1}{\Gamma(r) s^r} t^{r-1} e^{-t/s} \end{aligned}$$

- In either case, r is the "shape" parameter and a (or s) the "scale" or "rate" parameter.
- Important to check definition when using tables...
- ...although you will use rarely the $\gamma(a, r)$ directly.

The gamma distribution $\gamma(a, r)$ in R

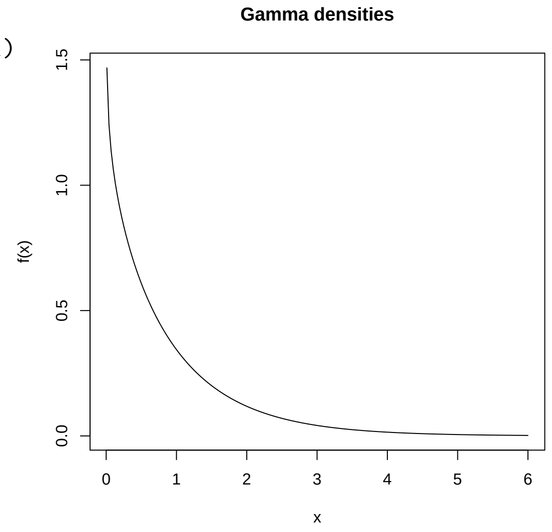
- Usual assortment of `(d,p,q,r)gamma)` functions.
- Sintax is, e.g. `dgamma(x, shape, rate, scale`

$$\begin{aligned} f_X(x) &= \frac{a^r}{\Gamma(r)} t^{r-1} e^{-at} \\ f_X(x) &= \frac{1}{\Gamma(r) s^r} t^{r-1} e^{-t/s} \end{aligned}$$

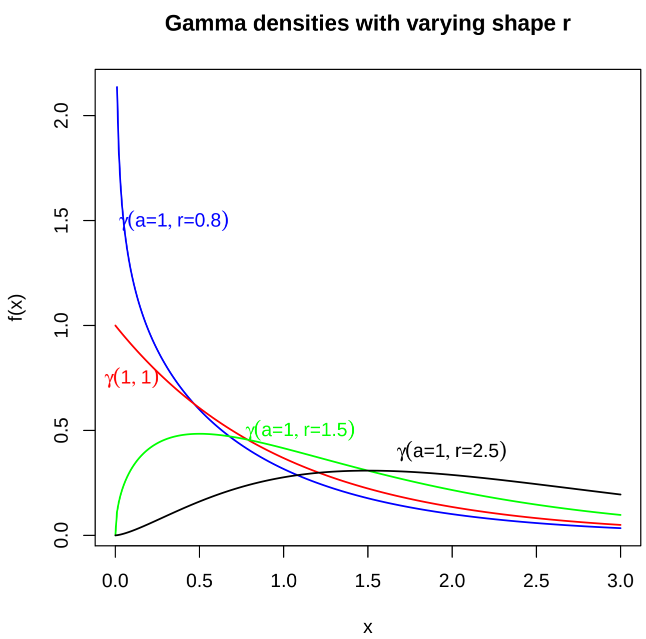
- In either case, r is the "shape" parameter and a the "rate" (or s is the "scale") or parameter.
- Only one of rate or scale needs to be specified.

What does the $\gamma(a, r)$ look like? (I)

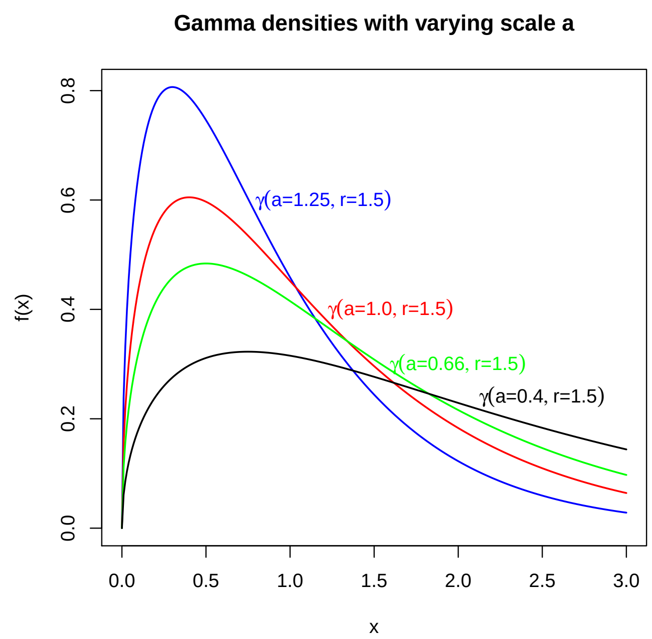
```
> gammar0.9 <- function(x) {
    dgamma(x,shape=0.9,scale=1)
    }
> curve(gammar0.9,from=0.01,
      to=6,n=200,
      ylab="f(x)",
      xlab="x",
      main="Gamma densities")
```



What does the $\gamma(a, r)$ look like? (II)



What does the $\gamma(a, r)$ look like? (III)



Moment generating function of the $\gamma(a, r)$ (I).

$$
\begin{aligned}
\varphi_X(u) &= E[e^{uX}] = \int_o^\infty \frac{a^r}{\Gamma(r)} x^{r-1} e^{-ax} e^{ux} dx \\
&= \frac{a^r}{\Gamma(r)} \int_o^\infty x^{r-1} e^{-(a-u)x} dx \\
&= \frac{a^r}{\Gamma(r)} \left[\frac{(a-u)^r}{\Gamma(r)} \right]^{-1} \\
&= \left(1 - \frac{u}{a}\right)^{-r}
\end{aligned}
$$

See how the integral went away?

It is equal to content within brackets in next-to-last expression.

Moment generating function of the $\gamma(a, r)$ (II).

- Let $X = X_1 + \ldots + X_n$ independent gamma random variables with equal scale parameter and respectively $r_1, \ldots, r_n$ as shape parameter. Then:

$$
\begin{aligned}
\varphi_X(u) &= \left(1 - \frac{u}{a}\right)^{-r_1} \cdots \left(1 - \frac{u}{a}\right)^{-r_n} \\
&= \left(1 - \frac{u}{a}\right)^{-r_1 + \ldots + r_n}
\end{aligned}
$$

  so X is $\gamma(a, r_1 + \ldots + r_n)$ distributed.
- The same does not hold if the scale parameters are not equal.

Mean and variance of $\gamma(a, r)$.

- Mean and variance are now easy to compute:

$$\begin{aligned}
[\varphi'_X(u)]_{u=0} &= \left[-r\left(1-\frac{u}{a}\right)^{-r-1}\left(-\frac{1}{a}\right)\right]_{u=0} \\
&= \frac{r}{a} \\
[\varphi''_X(u)]_{u=0} &= \left[r(r+1)\left(1-\frac{u}{a}\right)^{-r-2}\left(-\frac{1}{a}\right)^2\right]_{u=0} \\
&= \frac{r^2}{a^2}+\frac{r}{a^2}
\end{aligned}$$

Hence, $m = r/a$ and $\sigma^2 = \alpha_2 - (\alpha_1)^2 = r/a^2$.

- It can also be checked that the mode is at $\frac{r-1}{a}$ (or zero, in case $r < 1$ and monotone decreasing density).

How would you choose $\gamma(a, r)$ with mean 2 and variance 5?

Matching moments.

Exponential distribution $\exp(\lambda)$ (I)

- A very important particular case occurs when $r = 1$. Then,

$$\gamma(a, r = 1) = \frac{a^r}{\Gamma(r)} x^{r-1} e^{-ax} = ae^{-ax}$$

- Conventionally, a denoted by λ. Distribution called exponential, $\exp(\lambda)$.
- Alternative in terms of $\theta = 1/\lambda$:

$$f_X(x) = \lambda e^{-\lambda x} = \frac{1}{\theta} e^{-x/\theta}$$

- If we stick with the λ-parameterization, $m = 1/\lambda$ and $\sigma^2 = 1/\lambda^2$.
- Clearly, $F_X(x) = 1 - e^{-\lambda x}$.

Exponential distribution $\exp(\lambda)$ (II)

- The moment generating function comes straight from the $\gamma(a = \lambda, r = 1)$ general case:

$$\varphi_X(u) = \left(1 - \frac{u}{\lambda}\right)^{-1}$$

- With $f_X(x) = \lambda e^{-\lambda x}$ and $F_X(x) = 1 - e^{-\lambda x}$ no need of tables; however, still the usual R functions `{d,p,q,r}exp`.
- Sintax: `dexp(x,rate)` where `rate` is λ.

What if we sum n independent exponential variables with the same λ?

We get a variable distributed as $\gamma(\lambda, n)$.

Square-normal distribution

- If $X \sim N(0, 1)$, what is the distribution of $Y = X^2$?
- $F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$.
- Therefore $F_Y(y) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y})$, and

$$\begin{aligned}
f_Y(y) &= \phi(\sqrt{y}) \times \frac{1}{2\sqrt{y}} - \phi(-\sqrt{y}) \times \left(-\frac{1}{2\sqrt{y}}\right) \\
&= \phi(\sqrt{y})\frac{1}{\sqrt{y}} \\
&= \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-\frac{y}{2}} \qquad (y > 0)
\end{aligned}$$

What density is this a particular case of?

It is clearly a $\gamma(a = \frac{1}{2}, r = \frac{1}{2})$.

Things you can easily check:

- If Y is square-normal, $E[Y] = 1$.
  (Try it both ways, using the "gamma ancestry" of Y and the direct approach: remember $Y = X^2$ and $X \sim N(0,1)$).
- If Y is square-normal, its variance is 2.
- If $X_1, \ldots, X_n$ are i.i.d $N(0,1)$, then $Y = X_1^2 + \ldots + X_n^2$ is distributed as $\gamma(\frac{1}{2}, \frac{n}{2})$.
- Mimic the method used to derive the square-normal density to find the log-normal density, i.e., the density of Y such that $\log_e(Y)$ is normal.

The χ_n^2 distribution

- It is just the $\gamma(a = \frac{1}{2}, r = \frac{n}{2})$ obtained in last lecture…
- …or, if you prefer, the distribution of the sum of n independent $N(0,1)$ squared, each of which is $\gamma(a = \frac{1}{2}, r = \frac{1}{2})$
- As particular case of a $\gamma(a, r)$ we know:

$$m = n \qquad \sigma^2 = 2n \qquad \varphi_Y(u) = (1-2u)^{-\frac{n}{2}}$$
$$m = r/a \qquad \sigma^2 = r/a^2 \qquad \varphi_Y(u) = \left(1 - \frac{u}{a}\right)^{-r}$$

- n usually called "degrees of freedom".

What does it look like? (I)

- The density is,

$$f_X(x) = \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} x^{\frac{n}{2}-1} e^{-x/2}$$

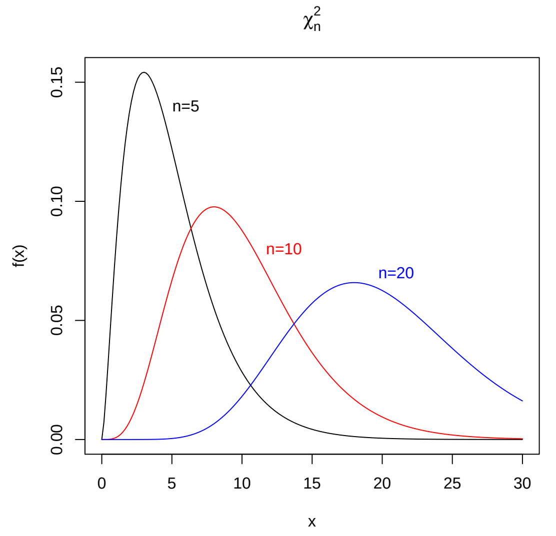
- As it is a $\gamma(a = \frac{1}{2}, r = \frac{n}{2})$, will be monotone decreasing for $r \leq 1$ ($\Rightarrow n \leq 2$).
- For $n > 2$ a single maximum and a long right tail (right-skewed).
- Becomes closer to symmetric as n grows.

What do you think the χ_n^2 converges to as $n \to \infty$?

$\chi_n^2 \xrightarrow{\text{d}} N(n, 2n)$ by the CLT.

What does it look like? (II)

```
> chisqn <- function(x) {
    dchisq(x,df=n)
  }
> n <- 5
> curve(chisqn,
  from=0.0,to=30,n=200,
  ylab="f(x)",xlab="x",
  main=expression(chi[n]^2))
> n <- 10
> curve(chisqn,from=0.0,col="red",
      to=30,n=200,add=TRUE)
> n <- 20
> curve(chisqn,from=0.0,col="blue",
      to=30,n=200,add=TRUE)
> text(6,0.14,"n=5")
> text(13,0.08,"n=10",col="red")
> text(21,0.07,"n=20",col="blue")
```



Non-central χ^2_n variables

- The ordinary or "central" χ^2_n is the sum of n independent $N(0,1)$ squared.
- If the squared normal variables have non-zero mean, we have instead the "non central" chi square.
- If $Y = X_1^2 + \ldots + X_n^2$ with $X_i \sim N(m_i, 1)$, then $Y \sim \chi^2_n(\delta)$ (the "non central" chi square).
- $\delta = m_1^2 + \ldots + m_n^2$ is the so-called "non-centrality parameter".
- Some tables/books define the non-centrality parameter as $\delta = \frac{1}{2}(m_1^2 + \ldots + m_n^2)$, so check.

χ^2_n in R

Usual set of functions: `{d,p,q,r}chisq`.

```
>     dchisq(15.3,12)
[1] 0.05196885
>     pchisq(15.3,12)
[1] 0.7745611
>     qchisq(0.99,12)
[1] 26.21697
>     qchisq(0.99,12,ncp=15)
[1] 52.15618
```

Snedecor's $\mathcal{F}_{m,n}$

- The ratio of two χ^2_m and χ^2_n independent of each other each divided by their degrees of freedom,

$$\frac{\chi^2_m/m}{\chi^2_n/n}$$

  follows a distribution named "Snedecor's $\mathcal{F}_{m,n}$" (after George W. Snedecor (1882 -1974)).
- Fairly complex density,

$$f_X(x) = \frac{m^{\frac{m}{2}} n^{\frac{n}{2}} \Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} x^{m/2-1}(n+mx)^{-(n+n)/2}$$

- For $n > 2$, $m = n/(n-2)$ if $n > 2$ and for $n > 4$,

$$\sigma^2 = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$$

Use of tables for $\mathcal{F}_{m,n}$

- Same as we did not need tables of $b(p,n)$ for $p > 0.5$, we can do with tables for the $\mathcal{F}_{m,n}$ for $\alpha < 0.5$ and obtain the rest indirectly.
- If $X \sim \mathcal{F}_{m,n}$, trick is to use

$$\begin{aligned} 1-\alpha = P(X < \mathcal{F}^\alpha_{m,n}) &= P\left(\frac{\chi^2_m/m}{\chi^2_n/n} < \mathcal{F}^\alpha_{m,n}\right) \\ &= P\left(\frac{\chi^2_n/n}{\chi^2_m/m} > \frac{1}{\mathcal{F}^\alpha_{m,n}}\right) \end{aligned}$$

  This shows,

$$\frac{1}{\mathcal{F}^\alpha_{m,n}} = \mathcal{F}^{1-\alpha}_{n,m}$$

Non-central versions of $\mathcal{F}_{m,n}$

- If the χ^2 in the numerator has non-centrality parameter δ, the resulting $\mathcal{F}_{m,n}$ is called non-central with the same non-centrality parameter.
- If both numerator and denominator are non-central χ^2, the ratio is a doubly non-central $\mathcal{F}_{m,n}$.
- Tables in general for only the ordinary or central case.

$\mathcal{F}_{m,n}$ in R

As usual, `{d,p,q,r}f` functions.

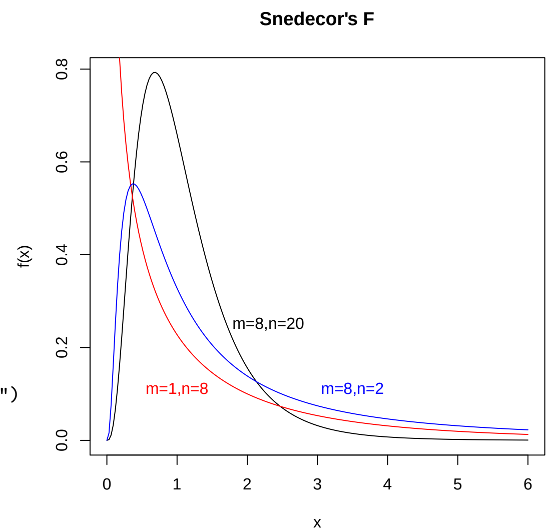
```
> pf(3.23, 5, 12)      # Prob left 3.23 in F(5;12)
[1] 0.9554027
> qf(0.95, 5, 12)      # Value leaving a tail of 0.05
[1] 3.105875
> qf(0.99, 5, 12)      # Id. for tail of 0.01
[1] 5.064343
> qf(0.99, 5, 12, 8)  # Id. for a non-central F
[1] 11.62582
>                      # with ncp=8
```

What does the $\mathcal{F}_{m,n}$ look like? (I)

- If n not too small, shape close to scaled χ^2_m.
- If both m and n large, closely concentrated around 1.
- Right-skewed.

What does the $\mathcal{F}_{m,n}$ look like? (II)

```
> sned <- function(x) {
     df(x,m,n)
   }
> m <- 8 ; n <- 20
> curve(sned,
   from=0.0,to=6,n=200,
   ylab="f(x)",xlab="x",
   main="Snedecor's F")
> m <- 1 ; n <- 8
> curve(sned,from=0.0,col="red",
       to=6,n=200,add=TRUE)
> m <- 8 ; n <- 2
> curve(sned,from=0.0,col="blue",
       to=6,n=200,add=TRUE)
> text(3.5,0.11,"m=8,n=2",col="blue")
> text(2.3,0.25,"m=8,n=20",col="black")
> text(1.0,0.11,"m=1,n=8",col="red")
```



Student's t_n distribution

- Distribution of the ratio of independent $N(0,1)$ and $\sqrt{\chi^2_n/n}$ random variables:

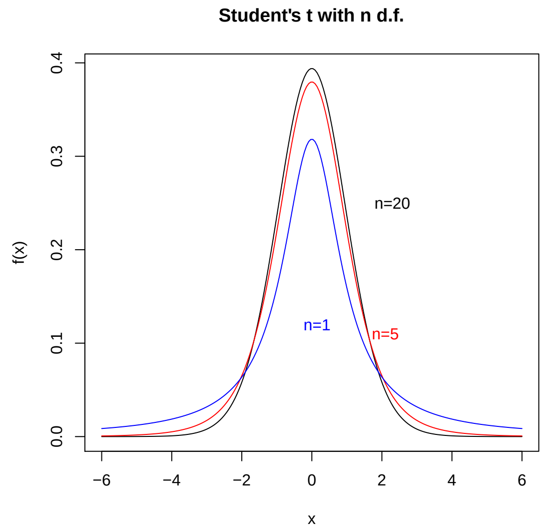
$$t_n = \frac{N(0,1)}{\sqrt{\chi^2_n/n}}$$

- Named after W. Gosset (1876-1937), who usually signed his work as "Student".
- Has density,

$$f_X(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{1}{2}(n+1)}$$

What does Student's t_n look like?

```
> tx <- function(x) {
    dt(x,n)
  }
> n <- 20
> curve(tx,
  from=-6,to=6,n=200,
  ylab="f(x)",xlab="x",
  main="Student's t with n d.f.")
> n <- 5
> curve(tx,from=-6,col="red",
      to=6,n=200,add=TRUE)
> n <- 1
> curve(tx,from=-6,col="blue",
      to=6,n=200,add=TRUE)
> text(0.15,0.12,"n=1",col="blue")
> text(2.3,0.25,"n=20",col="black")
> text(2.1,0.11,"n=5",col="red")
```



Moments of the t_n distribution

- Not all moments exist for all n.
- As an striking example, when $n = 1$,

$$\begin{aligned} f_X(x) &= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{x^2}{n}\right)^{-\frac{1}{2}(n+1)} \\ &= \frac{1}{\pi}\frac{1}{1+x^2} \end{aligned}$$

  is the Cauchy distribution, and has no mean!
- For greater n, higher order moments are non existent.

Some useful relationships

- $t_n^2 = \mathcal{F}_{1,n}$
- t_n approaches a $N(0,1)$ as $n \to \infty$.
- $\mathcal{F}_{m,n}$ approaches a χ^2_m as $n \to \infty$.
- If $X \sim \gamma(a, r)$ then $cX \sim \gamma(a/c, r)$.
- In particular, sum of exponentials, $= \gamma(\lambda, 1)$, can be turned into a χ^2_{2n} multiplying by a constant.