STATISTICS APPLIED TO BUSINESS ADMINISTRATION (BAD) - Second Year Academic Year 2019-20
STATISTICS APPLIED TO MARKETING (MD) - Second Year
STATISTICS APPLIED TO BUSINESS ADMINISTRATION (DD) - Third Year
First Call. May 27, 2020

INSTRUCTIONS

1. The exam contains multiple choice questions that must be answered in the orange code sheet we have provided you with, together with three exercises that must be answered in detail in separate sheets of paper.
2. To select an answer, all you need to do is mark in the orange code sheet, **filling the rectangle in which the selected answer is located appropriately**. Please make sure you know the answer you wish to mark before doing it. You can always cancel your mark by simply filling the rectangle below the answer you wish to cancel, and then fill the rectangle in which the new selected answer is located appropriately. Therefore, we advice you to first mark your selected answers in the exam and to use only the last ten minutes or so from the time assigned to the multiple choice questions-part of the exam to copy them into the code sheet.
3. In the multiple choice questions-part of the exam there is always **only one correct answer** for every question. **Every question correctly answered is worth 1 point, whereas each question incorrectly answered will not penalize your grade in any form. Therefore, you must answer all of the questions included in the exam**.
4. Each one of the exercises, A, B and C, must be answered in a separate sheet of paper. We will collect the different parts of the exam at the indicated times and in this order: first, we will collect the code sheet for the multiple choice questions part of the exam and, then, and in this order, exercises A, B and C will be collected.
5. The exam has six numbered sheets, going from 0.1 to 0.6. Please make sure that you have all sheets and contact your professor if this is not the case. There are different exam types. This exam is of type 0. Mark a 0 in the column labelled with I in your code sheet.
6. The maximum final grade for each of the parts of the exam (i.e., the multiple choice questions part and the exercises part) is 30 points. If you have participated in the on-going evaluation process, you will need to obtain 15 and 12 points in each part of the exam to pass it. Otherwise, 18 and 15 points in each part of the exam are required to pass it.
7. Please fill in your personal information in the appropriate places both in the code sheet and in the sheets provided for the exercises.

MULTIPLE CHOICE QUESTIONS (Time: 1 hour and 35 minutes)

1. FREE-QUESTION. The capital of Spain is:

   (A) Paris (B) Sebastopol (C) Madrid (D) London (E) Pekin

Questions 2 to 4 refer to the following exercise:

The probability that a client enters a specific bank branch and requests information about mortgages is equal to 25%. It is known that 20 clients enter that bank branch every day. We assume independence between the different clients entering that bank branch.

2. The probability that, in a given day, 4 clients request information about mortgages is:

   (A) 0.0898 (B) 0.0669 (C) 0.2446 (D) 0.2023 (E) 0.1897

3. The probability that, in a given day, 9 clients **do not request** information about mortgages is:

   (A) 0.0609 (B) 0.0099 (C) 0.0030 (D) 0.0005 (E) 0.0909

4. The approximate probability that, in a 10-day period, at least 39 clients request information about mortgages is:

   (A) 0.3821 (B) 0.0301 (C) 0.9699 (D) 0.6179 (E) 0.9474

Questions 5 and 6 refer to the following exercise:

Let Z be a r.v. following a binomial $b(p, n)$ distribution with mean $\mathrm{E}(Z) = 4.50$ and variance $\mathrm{Var}(Z) = 3.15$.

5. $P(Z > 8)$ is:

   (A) 0.0152 (B) 0.0500 (C) 0.9500 (D) 0.0037 (E) 0.9848

6. $P(2 \leq Z \leq 8)$ is:

   (A) 0.9495 (B) 0.9848 (C) 0.8695 (D) 0.9610 (E) 0.8580

Questions 7 to 10 refer to the following exercise:

The number of individuals arriving at a coffee shop in a one-minute period follows a Poisson distribution such that $P(X = 3) = \frac{1}{2}\, P(X = 2)$. We assume independence between the clients arriving at the different one-minute periods.

7. The probability that, in a given minute, 4 individuals arrive at the coffee shop is:

   (A) 0.0153 (B) 0.2824 (C) 0.0471 (D) 0.1412 (E) 0.0902

8. The most likely number of individuals that would arrive at the coffee shop in a given minute is:

   (A) 1 (B) 1.5 (C) 0 (D) 2.5 (E) 2

9. The probability that, in a two-minute period, at least 4 individuals arrive at the coffee shop is:

   (A) 0.3528 (B) 0.8571 (C) 0.6472 (D) 0.1429 (E) 0.1847

10. The approximate probability that, in a 60-minute period, at most 98 individuals arrive at the coffee shop is:

(A) 0.8413 (B) 0.1587 (C) 0.8159 (D) 0.8365 (E) 0.1841

Questions 11 and 12 refer to the following exercise:

Let $X_1, \ldots, X_4$ be independent and identically distributed r.v. having a $\gamma(0.50, 0.75)$ distribution.

11. If we define the r.v. $Y = X_1 + X_2 + X_3 + X_4$, the value of k such that $P(Y > k) = 0.90$ is:

(A) 2.20 (B) 1.06 (C) 0.584 (D) 4.17 (E) 10.6

12. The distribution of the r.v. $Z = 2X_1 + 2X_4$ is:

(A) $\gamma(0.25, 0.75)$ (B) $\gamma(0.50, 0.75)$ (C) $\gamma(0.50, 0.50)$ (D) $\gamma(0.50, 1.50)$ (E) $\gamma(0.25, 1.50)$

Questions 13 to 15 refer to the following exercise:

Let X, Y, Z and V be four independent r.v. such that their distributions are as follows: $X \in N(0, \sigma^2 = 1)$, $Y \in N(0, \sigma^2 = 8)$, $Z \in \exp(\frac{1}{2})$ and $V \in \gamma(\frac{1}{2}, 5)$.

13. If we define the r.v. $W_1 = Z + V$, the value of k such that $P(k < W_1 < 21) = 0.70$ is:

(A) 6.30 (B) 5.90 (C) 9.30 (D) 8.44 (E) 6.74

14. If we define the r.v. $W_2 = \dfrac{\sqrt{2}(X+Y)}{3\sqrt{Z}}$, the value of k such that $P(W_2 < k) = 0.20$ is:

(A) -1.061 (B) -1.890 (C) -3.080 (D) 3.080 (E) 1.061

15. If we define the r.v. $W_3 = \frac{5Y^2}{4V}$, then $P(W_3 < 4.96)$ is:

(A) 0.95 (B) 0.01 (C) 0.05 (D) 0.90 (E) 0.99

Questions 16 and 17 refer to the following exercise:

The lifetime for the light bulbs from a given firm, in thousands of hours, follows a probability distribution that depends on the parameters θ and γ. It is known that $\mathrm{E}(X) = \theta\gamma$ and that $\mathrm{E}(X^2) = \theta\gamma(1+\theta)$. In order to estimate these parameters, a r.s. of four of those light bulbs was taken providing the following lifetimes: 4.1, 4.5, 3.9 and 5 thousands of hours, respectively.

16. The method of moments estimate of θ is:

(A) $\hat{\theta}_{\mathrm{MM}} = 9.13$ (B) $\hat{\theta}_{\mathrm{MM}} = 1.54$ (C) $\hat{\theta}_{\mathrm{MM}} = 4.42$ (D) $\hat{\theta}_{\mathrm{MM}} = 3.42$ (E) $\hat{\theta}_{\mathrm{MM}} = 4.82$

17. The method of moments estimate of γ is:

(A) $\hat{\gamma}_{\mathrm{MM}} = 0.27$ (B) $\hat{\gamma}_{\mathrm{MM}} = 1.63$ (C) $\hat{\gamma}_{\mathrm{MM}} = 1.28$ (D) $\hat{\gamma}_{\mathrm{MM}} = 0.20$ (E) $\hat{\gamma}_{\mathrm{MM}} = 4.38$

Questions 18 and 19 refer to the following exercise:

Let X be a discrete r.v. with probability mass function given by:

$$P(X=0)=2\theta;\quad P(X=1)=\frac{1}{2}-\theta;\quad P(X=-1)=\frac{1}{2}-\theta.$$

In order to estimate the parameter θ a r.s. of size n has been taken, providing three zeroes.

18. The maximum likelihood estimate of θ is:

(A) $\dfrac{n-3}{2n}$ (B) $\dfrac{3}{2n}$ (C) $\dfrac{1}{n}$ (D) $\dfrac{n-3}{n}$ (E) $\dfrac{3}{n}$

19. The method of moments estimate of θ is:

(A) $\dfrac{3}{n}$ (B) $\dfrac{3}{2n}$ (C) $\dfrac{n-3}{2n}$ (D) $\dfrac{n-3}{n}$ (E) $\dfrac{1}{n}$

Questions 20 to 22 refer to the following exercise:

Let X be a r.v. having a Poisson distribution with parameter θ. In order to estimate the parameter θ, a r.s. of size n, $X_1,\dots,X_n$, has been taken, and $\hat{\theta}=(X_1+3X_2+\ldots+3X_{n-1}+4X_n)/3n$ is proposed as an estimator of θ.

20. The bias of the proposed estimator is:

(A) 0 (B) $-\dfrac{\theta}{n}$ (C) $\dfrac{1}{3n}$ (D) $\dfrac{\theta}{3n}$ (E) $-\dfrac{\theta}{3n}$

21. The proposed estimator is:

(A) Biased and asymptotically unbiased (B) – (C) Unbiased

(D) Biased and asymptotically biased (E) Unbiased an asymptotically biased

22. The variance of the proposed estimator is:

(A) $\dfrac{\theta(9n-1)}{9n^2}$ (B) $\dfrac{\theta}{3n}$ (C) $\dfrac{\theta}{9n^2}$ (D) $\dfrac{\theta(3n-1)}{9n^2}$ (E) $\dfrac{\theta}{n^2}$

Questions 23 to 25 refer to the following exercise:

Let X be a r.v. with probability density function given by:

$$f(x,\theta)=\theta^3\, x^{\theta^3-1},\quad 0<x<1,\quad \theta>0$$

In order to test the null hypothesis $H_0:\theta=1$ against the alternative hypothesis $H_1:\theta=2$, a r.s. of size $n=1$, X_1, has been taken.

23. The most powerful critical region for the test statistic X_1 is of the form:

(A) (k_1,k_2) (B) $[k_1,1)$ (C) $(0,k_1]$ (D) $(k_1,k_2)^c$ (E) All false

24. At the $\alpha=0.10$ significance level, the most powerful test will reject the null hypothesis if:

(A) $X_1\geq 0.90$ (B) $X_1\in(0.10,0.90)$ (C) $X_1\leq 0.01$ (D) $X_1\geq 0.729$ (E) $X_1\leq 0.10$

25. The power for this test is:

(A) 0.1900 (B) 0.5695 (C) 0.8100 (D) 0.5217 (E) 0.4305

26. Let X be a r.v. having a Poisson $\mathcal{P}(\lambda)$ distribution. In order to test the null hypothesis $H_0 : \lambda = 1$ against the alternative hypothesis $H_1 : \lambda = 1.8$, a r.s. of size $n = 6$ has been taken, and $Z = \sum_{i=1}^{6} X_i$ is used as test statistic. At the $\alpha = 0.15$ significance level, the null hypothesis is rejected if:

(A) $Z \geq 10$ (B) $Z \leq 9$ (C) $Z \geq 8$ (D) $Z \geq 9$ (E) $Z \leq 10$

Questions 27 and 28 refer to the following exercise:

A car manufacturer claims that one of the models his/her firm develops (turbo model: T) consumes less fuel than another model (sport model : S) developed by a competitor. In order to test his/her hypothesis, two r.s. of 21 cars each, were taken and their fuel consumption per 100 Km., in liters, was recorded. Results provided by the sample were as follows: $\overline{x}_T = 6.2$, $\overline{x}_S = 5.9$, $s_T^2 = 1.14$, $s_S^2 = 1.05$. We assume that the fuel consumption distributions for both models are independent from each other, normally distributed and having a common variance.

27. The firm wishes to test the null hypothesis that the variance of the fuel consumption for the turbo model is equal to that of the sport model. At the $\alpha = 0.10$ significance level, the decision of the firm will be:

(A) Do not reject the null hypothesis (B) – (C) Reject the null hypothesis

(D) – (E) –

28. The **95% confidence interval** for the difference of the mean fuel consumption for the two models above, $(m_T - m_S)$ is, approximately:

(A) $(0.47, 0.97)$ (B) $(0.08, 52)$ (C) $(0.26, 0.86)$ (D) $(-0.26, 0.86)$ (E) $(-0.37, 0.97)$

Questions 29 and 30 refer to the following exercise:

Let X be a r.v. following a binary $b(p)$ distribution. In order to test the null hypothesis $H_0 : p = 0.40$ against the alternative hypothesis $H_1 : p > 0.40$, a r.s. of size $n = 10$ has been taken, and $Z = \sum_{i=1}^{10} X_i$ is used as test statistic.

29. At the $\alpha = 0.10$ significance level, the null hypothesis is rejected if:

(A) $Z \leq 7$ (B) $Z \leq 6$ (C) $Z \geq 7$ (D) $Z \geq 6$ (E) $Z \geq 8$

30. The probability of type II error for the critical region above and $p = 0.70$ is:

(A) 0.3504 (B) 0.9894 (C) 0.1503 (D) 0.8497 (E) 0.6496

EXERCISES (Time: 65 minutes)

A. (10 points, 20 minutes)

In a recent study based on a random sample of 300 car accidents, the compiled information was classified according to the size of the car.

	Small size	Medium size	Large size
Some deaths	42	35	20
No deaths	78	65	60

With this information, can we claim that mortality caused by accidents depends on the size of the car? Use a significance level $\alpha = 0.05$.

B. (10 points, 20 minutes) Let X be a r.v. having a $\gamma(a, 4)$ distribution. That is, having the following probability density function:

$$f_X(x; a) = \frac{a^4}{6} x^3 e^{-ax}, \ x > 0, \ a > 0$$

We wish to test the null hypothesis $H_0 : a = \frac{1}{2}$ against the alternative hypothesis $H_1 : a = \frac{1}{4}$. In order to do to, a r.s. of size $n = 3$, X_1, X_2, X_3, has been taken.

i) Obtain the form of the most powerful critical region for this test. In order to do so, you **must show**, as part of your solution to this exercise, that we reject the null hypothesis if $S = \sum_{i=1}^{n} X_i \geq C, \ C > 0$.

ii) Under the null hypothesis, what is the distribution of the test statistic?

iii) At the $\alpha = 0.10$ significance level, obtain the specific most powerful critical region for this test.

C. (10 points, 25 minutes) Let $X_1, X_2, \ldots, X_n$ be independent r.v. such that, for each i, X_i follows a Poisson distribution with parameter $k_i\theta$. That is, $X_i \in \mathcal{P}(k_i\theta), \ i = 1, \ldots, n$, where the k_i's are known positive constants and $\theta > 0$. That is,

$$P(x_i) = \frac{k_i^{x_i}}{x_i!} \ e^{-k_i\theta} \ \theta^{x_i}, \ \ x_i = 0, 1, 2, \cdots, \ \ k_i > 0, \ \theta > 0.$$

i) Obtain, **providing all relevant information**, the maximum likelihood estimator of θ.

ii) Is it unbiased? Is it consistent? **Remark**: You can assume that $\lim_{n\to\infty} \sum_{i=1}^{n} k_i = +\infty$. In addition, **make sure that you provide all of the relevant information** to support your answers to these questions.

SOLUTIONS TO MULTIPLE CHOICE QUESTIONS (exam type 0)

1: C	11: A	21: A
2: E	12: E	22: A
3: C	13: D	23: B
4: C	14: A	24: A
5: A	15: A	25: B
6: A	16: D	26: A
7: C	17: C	27: A
8: A	18: B	28: E
9: A	19: B	29: C
10: C	20: E	30: A

SOLUTIONS TO EXERCISES

Exercise A

	Small size	Medium size	Large size
Some deaths	42	35	20
No deaths	78	65	60

To test if the mortality caused by accidents depends on the size of the car, we have to carry out a test of independence, where the null hypothesis states that the variables mortality and size of the car are independent variables, whereas the alternative hypothesis states that they are not independent variables.

The estimated probabilities $\hat{p}_i$ and $\hat{p}_j$ are computed from the information included in the table above:

$$\hat{p}_{\text{SD}} = \frac{(42+35+20)}{300} = \frac{97}{300} = 0.3233 \qquad \hat{p}_{\text{ND}} = \frac{(78+65+60)}{300} = \frac{203}{300} = 0.6767$$

$$\hat{p}_{\text{S}} = \frac{(42+78)}{300} = \frac{120}{300} = 0.4000 \qquad \hat{p}_{\text{M}} = \frac{(35+65)}{300} = \frac{100}{300} = 0.3333 \qquad \hat{p}_{\text{L}} = \frac{(20+60)}{300} = \frac{80}{300} = 0.2667$$

We are now able to build the following table:

	n_{ij}	$\hat{p}_{ij} = \hat{p}_i\hat{p}_j$	$n\hat{p}_{ij}$	$\frac{(n_{ij}-n\hat{p}_{ij})^2}{n\hat{p}_{ij}}$
(SD,S)	42	$0.3233 \times 0.4000 = 0.1293$	38.79	0.2656
(SD,M)	35	$0.3233 \times 0.3333 = 0.1078$	32.34	0.2188
(SD,L)	20	$0.3233 \times 0.2667 = 0.0862$	25.86	1.3279
(ND,S)	78	$0.6767 \times 0.4000 = 0.2707$	81.21	0.1269
(ND,M)	65	$0.6767 \times 0.3333 = 0.2256$	67.68	0.1061
(ND,L)	60	$0.6767 \times 0.2667 = 0.1804$	54.12	0.6388
Total	$n = 300$	1	$n = 300$	$z = 2.6841$

Under the null hypothesis of independence, the test statistic $\sum_{i,j} \frac{(n_{ij}-n\hat{p}_{ij})^2}{n\hat{p}_{ij}}$ converges to a $\chi^2_{(I-1)(J-1)}$ distribution, where I is the number of classes in which the first variable is divided and J is the number of classes in which the second variable is divided. That is, we then have that $I = 2$ and $J = 3$.

In this case:

$$z = 2.6841 < 5.99 = \chi^2_{(2-1)(3-1)\,,\,0.05} = \chi^2_{2,0.05},$$

so that, at an approximate 5% significance level, we do not reject the null hypothesis of independence between the variables mortality and size of the car. That is, we do not reject that accidents' mortality does not depend on the size of the car.

Exercise B

i) X is a r.v. with probability density function given by:

$$f_X(x;a) = \frac{a^4}{6}x^3e^{-ax}, x > 0, a > 0,$$

and we wish to the null hypothesis that $a = \frac{1}{2}$ against the alternative hypothesis that $a = \frac{1}{4}$. That is, $H_0 : a = \frac{1}{2}$, $H_1 : a = \frac{1}{4}$.

For a given significance level, the most powerful critical region is obtained from the likelihood ratio test. That is,

$$\frac{L(\vec{x}|H_0)}{L(\vec{x}|H_1)} \leq k$$

In this case, the likelihood function is:

$$L(\vec{x}, a) = \left[\frac{a^4}{6}x_1^3e^{-ax_1}\right]\ldots\left[\frac{a^4}{6}x_n^3e^{-ax_n}\right] = \frac{a^{4n}}{6^n}\left(\Pi_{i=1}^n x_i^3\right)e^{-a\sum_{i=1}^n x_i},$$

and the critical region will then be obtained from:

$$\frac{L(\vec{x}|H_0)}{L(\vec{x}|H_1)} = \frac{\frac{\left(\frac{1}{2}\right)^{4n}}{6^n}\left(\Pi_{i=1}^n x_i\right)^3 e^{-\frac{1}{2}\left(\sum_{i=1}^n x_i\right)}}{\frac{\left(\frac{1}{4}\right)^{4n}}{6^n}\left(\Pi_{i=1}^n x_i\right)^3 e^{-\frac{1}{4}\left(\sum_{i=1}^n x_i\right)}} \leq k$$

$$\frac{L(\vec{x}|H_0)}{L(\vec{x}|H_1)} = \frac{\left(\frac{1}{2}\right)^{4n} e^{-\frac{1}{2}\left(\sum_{i=1}^n x_i\right)}}{\left(\frac{1}{4}\right)^{4n} e^{-\frac{1}{4}\left(\sum_{i=1}^n x_i\right)}} = 2^{4n}e^{-\frac{1}{4}\sum_{i=1}^n x_i} \leq k_1$$

$$\Longrightarrow -\frac{1}{4}\sum_{i=1}^n x_i \leq k_2 \Longrightarrow \text{Rechazar } H_0 \text{ si } S = \sum_{i=1}^n X_i \geq C$$

Therefore, the general form of the most powerful critical region for the test statistic $S = \sum_{i=1}^n X_i$ is $\text{CR} = [C, +\infty)$.

ii) Given that, under H_0, $X_i \in \gamma\left(\frac{1}{2}, 4\right)$, and that $n = 3$, if we use the convolution property, we have that,

$$S = \sum_{i=1}^3 X_i \in \gamma\left(\frac{1}{2}, 12\right) \equiv \chi^2_{24}$$

iii) We wish to find the value of C in the specific case where $\alpha = 10\%$. That is,

$$\alpha = 0.10 = P(S \geq C|H_0)$$

Given that, under the null hypothesis, $S \in \chi^2_{24|}$, the value of C for which this condition holds, is $C = \chi^2_{30|0.10} = 33.2$. Therefore, the specific form of the critical region for this significance level will be:

$$\text{CR} = [33.2, +\infty)$$

Exercise C

Given that $X_i \in \mathcal{P}(k_i\theta)$, we will have that

$$P(x_i) = \frac{k_i^{x_i}}{x_i!} \ \ e^{-k_i\theta} \ \ \theta^{x_i}, \quad x_i = 0, 1, 2, \cdots, \quad k_i > 0, \ \theta > 0.$$

i) In this way, the likelihood function will be given by

$$L(\theta) = P(x_1; k_1\theta) \cdots P(x_n; k_n\theta)$$

$$L(\theta) = \left[\frac{k_1^{x_1}}{x_1!} \ \ e^{-k_1\theta} \ \ \theta^{x_1}\right] \cdots \left[\frac{k_n^{x_n}}{x_n!} \ \ e^{-k_n\theta} \ \ \theta^{x_n}\right]$$

$$L(\theta) = \frac{\Pi_{i=1}^n k_i^{x_i}}{\Pi_{i=1}^n x_i!} \ \ e^{-\theta \sum_{i=1}^n k_i} \ \ \theta^{\sum_{i=1}^n x_i}$$

The maximum likelihood estimator of θ is the value that maximizes the likelihood function or, equivalently, its natural logarithm:

$$\ln L(\theta) = \sum_{i=1}^n x_i \ln k_i - \sum_{i=1}^n \ln x_i! - \theta \sum_{i=1}^n k_i + (\ln \theta) \sum_{i=1}^n x_i$$

If we take derivatives with respect to θ and equate this to zero, we will have that:

$$\frac{\partial \ln L(\theta)}{\partial \theta} = 0,$$

so that

$$-\sum_{i=1}^n k_i + \frac{\sum_{i=1}^n x_i}{\theta} = 0 \Longrightarrow \hat{\theta}_{\mathrm{ML}} = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n k_i}$$

ii) This estimator will be unbiased if $\mathrm{E}(\hat{\theta}_{\mathrm{ML}}) = \theta$.

$$\mathrm{E}(\hat{\theta}_{\mathrm{ML}}) = \frac{1}{\sum_{i=1}^n k_i} \mathrm{E}\left(\sum_{i=1}^n X_i\right) = \frac{1}{\sum_{i=1}^n k_i} \sum_{i=1}^n \mathrm{E}(X_i) = \frac{1}{\sum_{i=1}^n k_i} \sum_{i=1}^n k_i\theta = \frac{\theta\left(\sum_{i=1}^n k_i\right)}{\left(\sum_{i=1}^n k_i\right)} = \theta$$

Therefore, $\hat{\theta}_{\mathrm{ML}}$ in an unbiased estimator of θ. To check if the estimator $\hat{\theta}_{\mathrm{ML}}$ is consistent, we have to verify if the two sufficient conditions stated below hold:

a) $\lim_{n\to\infty} \mathrm{E}(\hat{\theta}_{\mathrm{ML}}) = \theta$

b) $\lim_{n\to\infty} \mathrm{Var}(\hat{\theta}_{\mathrm{ML}}) = 0$

Given that $\hat{\theta}_{\mathrm{ML}}$ is an unbiased estimator of θ, condition a) holds. In addition, we have that

$$\lim_{n\to\infty} \mathrm{Var}(\hat{\theta}_{\mathrm{ML}}) = \lim_{n\to\infty} \frac{1}{\left(\sum_{i=1}^n k_i\right)^2} \sum_{i=1}^n \mathrm{Var}(X_i)$$

$$\lim_{n\to\infty} \mathrm{Var}(\hat{\theta}_{\mathrm{ML}}) = \lim_{n\to\infty} \frac{1}{\left(\sum_{i=1}^n k_i\right)^2} \sum_{i=1}^n k_i\theta = \lim_{n\to\infty} \frac{\left(\sum_{i=1}^n k_i\right)}{\left(\sum_{i=1}^n k_i\right)^2} \ \ \theta = \lim_{n\to\infty} \frac{\theta}{\sum_{i=1}^n k_i} = 0,$$

given that, according to the conditions provided in this exercise, we can assume that $\lim_{n\to\infty} \sum_{i=1}^n k_i = +\infty$. In this way, both of the sufficient conditions hold and, thus, $\hat{\theta}_{\mathrm{ML}}$ is a consistent estimator of θ.