INSTRUCTIONS

1. The exam contains multiple choice questions that must be answered in the orange code sheet we have provided you with, together with three exercises that must be answered in detail in separate sheets of paper.
2. To select an answer, all you need to do is mark in the orange code sheet, **filling the rectangle over which the selected answer is located appropriately**. Please make sure you know the answer you wish to mark before doing it. Even though you can always erase your mark if you have used a pencil (number 2 or similar), any mark that has not been completely erased could be read by the machine. Therefore, we advice you to first mark your selected answers in the exam and to use only the last ten minutes or so from the time assigned to the multiple choice questions-part of the exam to copy them into the code sheet.
3. In the multiple choice questions-part of the exam there is always **only one correct answer** for every question. Every question correctly answered is worth 1 point, while each question incorrectly answered will penalize your grade by 0.2 points. Questions that have not been answered do not penalize your grade in any form.
4. Each one of the exercises, A, B and C, must be answered in a separate sheet of paper. We will collect the different parts of the exam at the indicated times and in this order: first, we will collect the code sheet for the multiple choice questions part of the exam and, then, and in this order, exercises A, B and C will be collected.
5. The exam has six numbered sheets, going from 0.1 to 0.6. Please make sure that you have all sheets and contact your professor if this is not the case. There are different exam types. This exam is of type 0. Mark a 0 in the column labelled with I in your code sheet.
6. The maximum final grade for each of the parts of the exam (i.e., the multiple choice questions part and the exercises part) is 30 points. If you have participated in the on-going evaluation process, you will need to obtain 12 points in each part of the exam to pass it. Otherwise, 15 points points in each part of the exam are required to pass it.
7. Please fill in your personal information in the appropriate places both in the code sheet and in the sheets provided for the exercises.

MULTIPLE CHOICE QUESTIONS (Time: 1 hour and 30 minutes)

1. FREE-QUESTION. The capital of Spain is:

   (A) Paris (B) Sebastopol (C) Madrid (D) London (E) Pekin

Questions 2 to 4 refer to the following exercise:

The probability that students belonging to a randomly selected high school fail their University admission exam is 0.12. We assume independence between the different students belonging to this high school.

2. If a random sample of 12 students from this high school is taken, the probability that exactly 3 of them fail their University admission exam is:

   (A) 0.2157 (B) 0.0080 (C) 0.0369 (D) 0.2647 (E) 0.1203

3. In the same sample of 12 students from the previous question, the probability that 9 of them pass their University admission exam is:

   (A) 0.1203 (B) 0.0080 (C) 0.2157 (D) 0.2647 (E) 0.0369

4. If we now take a random sample of 200 students from this high school, the approximate probability that less than 28 of them fail their University admission exam is:

   (A) 0.2236 (B) 0.5636 (C) 0.9792 (D) 0.4364 (E) 0.7764

Questions 5 and 6 refer to the following exercise:

Let Z be a r.v. such that it follows a $b(p, n)$ binomial distribution, with $n = 10$ and mean equal to 8.5.

5. The probability that this r.v. takes the value 10, $P(Z = 10)$, is equal to:

   (A) 0.1298 (B) 0.2759 (C) 0.1969 (D) 1 (E) 0.3474

6. The characteristic function of the r.v. Z is:

(A) $\psi_Z(u) = (0.15 + 0.85e^{iu})^{10}$ (B) $\psi_Z(u) = e^{10(e^{iu}-1)}$ (C) $\psi_Z(u) = (0.85 + 0.15e^{iu})^{10}$

(D) $\psi_Z(u) = (0.15 + 0.85e^{iu})$ (E) $\psi_Z(u) = e^{8.5(e^{iu}-1)}$

7. Let X be a r.v. having a Poisson distribution such that $P(X = 3) = 0.2125$ and $P(X = 4) = 0.1912$. The value of the mode of the r.v. X is:

   (A) 1 (B) 4 (C) 2 (D) 3 (E) 5

Questions 8 to 10 refer to the following exercise:

Let $X_1, \ldots, X_{40}$ be independent and identically distributed r.v. having a Poisson distribution with mean equal to 4.

8. $P(X_{24} > 5)$ is:

   (A) 0.7851 (B) 0.1107 (C) 0.8893 (D) 0.3712 (E) 0.2149

9. $P(3 \leq X_1 < 8)$ is:

(A) 0.5154 (B) 0.3498 (C) 0.7405 (D) 0.3200 (E) 0.7108

10. If we define $Y = \sum_{i=1}^{40} X_i$, then $P(Y \geq 140)$ is approximately equal to:

(A) 0.0526 (B) 0.4522 (C) 0.9474 (D) 0.1281 (E) 0.5478

Questions 11 to 14 refer to the following exercise:

Let X, Y, Z and W be independent r.v. with distributions such that: $X \in N(0,1)$, $Y \in \chi^2_6$, $Z \in \chi^2_{12}$ and $W \in t_{15}$.

11. The probability that the r.v. $V_1 = X^2 + Y$ takes on values larger than 4.25 is:

(A) 0.95 (B) 0.05 (C) 0.50 (D) 0.75 (E) 0.25

12. We define the r.v. V as $V_2 = \frac{7Z}{12V_1}$. The value of k such that $P(V_2 > k) = 0.99$ is:

(A) 4.64 (B) 0.15 (C) 6.47 (D) 0.40 (E) 0.22

13. $P(-0.866 < W < 1.34)$ is:

(A) 0.30 (B) 0.50 (C) 0.70 (D) 0.80 (E) 0.90

14. The characteristic function of the r.v. $V_3 = Y + Z$ is:

(A) $\psi_{V_3}(u) = (1-2iu)^{-18}$ (B) $\psi_{V_3}(u) = \left(1 - \frac{iu}{18}\right)^{-\frac{1}{2}}$ (C) $\psi_{V_3}(u) = (1-iu)^{-9}$
(D) $\psi_{V_3}(u) = (1-iu)^{-18}$ (E) $\psi_{V_3}(u) = (1-2iu)^{-9}$

Questions 15 and 16 refer to the following exercise:

Let X be a discrete r.v. with probability mass function given by:

$$P(X=-1) = \frac{3\theta}{2} \qquad P(X=0) = \frac{3\theta}{2} \qquad P(X=1) = 1 - 3\theta$$

In order to estimate the parameter θ, a r.s. of size n, $X_1, X_2, \ldots, X_n$, has been taken.

15. The method of moments estimator of θ is:

(A) $\overline{X}$ (B) $\dfrac{2-\overline{X}}{9}$ (C) $\dfrac{2(1-\overline{X})}{9}$ (D) $\dfrac{9}{2(1-\overline{X})}$ (E) $\dfrac{1-2\overline{X}}{9}$

16. In order to be able to obtain an estimate of the parameter θ, a random sample of size $n = 10$ has been taken providing the following results: -1, -1, -1, -1, 0, 0, 1, 1, 1, 1. A maximum likelihood estimate of θ is equal to:

(A) 0.20 (B) 0.17 (C) 0.33 (D) 0.25 (E) 0.80

Questions 17 to 20 refer to the following exercise:

Let X be a r.v. having a gamma $\gamma\left(\frac{1}{\theta}, 3\right)$ distribution, so that its probability density function is given by:

$$f(x,\theta) = \begin{cases} \dfrac{x^2}{2\theta^3}\, e^{-\frac{x}{\theta}} & \text{para } x \geq 0,\ \theta > 0; \\ 0 & \text{otherwise,} \end{cases}$$

We wish to estimate the parameter θ. In order to do so, a r.s. of size n, $X_1, X_2, \ldots, X_n$, is taken.

17. The maximum likelihood estimator of θ, $\hat{\theta}_{\mathrm{ML}}$, will be:

(A) $3\overline{X}$ (B) $\frac{\overline{X}}{3}$ (C) $\overline{X}$ (D) $3n\overline{X}$ (E) $\frac{\overline{X}}{3n}$

18. The method of moments estimator of θ, $\hat{\theta}_{\mathrm{MM}}$, will be:

(A) $\overline{X}$ (B) $\frac{\overline{X}}{3}$ (C) $\frac{\overline{X}}{3n}$ (D) $3n\overline{X}$ (E) $3\overline{X}$

19. The bias for the method of moments estimator of θ is:

(A) θ (B) 3θ (C) 0 (D) $\dfrac{\theta}{3}$ (E) $\dfrac{\theta}{n}$

20. The mean square error for the method of moments estimator of θ is:

(A) $\dfrac{\theta^2}{n}$ (B) $\dfrac{\theta^2}{9n}$ (C) θ^2 (D) $\dfrac{1}{9n}$ (E) $\dfrac{\theta^2}{3n}$

Questions 21 to 23 refer to the following exercise:

Let X be a r.v. with probability density function given by:

$$f(x, \theta) = \theta^2 \; x^{\theta^2 - 1}, \quad 0 < x < 1, \quad \theta > 0$$

Based on a r.s. of size $n = 1$, X_1, we wish to test the null hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_1 : \theta = 2$.

21. The most powerful critical region for this test and for the test statistic X_1 will be of the form:

(A) $(0, C]$ (B) $[C_1, C_2]^C$ (C) $[C, 1)$ (D) $[C_1, C_2]$ (E) All false

22. At the $\alpha = 0.10$ significance level, the most powerful critical region for the test statistic X_1 will be:

(A) $(0, 0.90]$ (B) $[0.10, 0.90]^C$ (C) All false (D) $[0.10, 0.90]$ (E) $[0.90, 1)$

23. For the same significance level, the probability of type II error for this test is approximately equal to:

(A) 0.66 (B) 0.34 (C) 0.73 (D) 0.90 (E) 0.27

Questions 24 and 25 refer to the following exercise:

Let X be a r.v. having a binary distribution with parameter p, $b(p)$, $0 < p < 1$. In order to test the null hypothesis $H_0 : p = 0.30$ against the alternative hypothesis $H_1 : p > 0.30$, a r.s. of size $n = 10$ has been taken, and the use of the the test statistic $Z = \sum_{i=1}^{10} X_i$ is proposed.

24. At the $\alpha = 0.10$ significance level, the null hypothesis is rejected if:

(A) $Z \geq 6$ (B) $Z \geq 5$ (C) $Z \leq 5$ (D) $Z \geq 7$ (E) $Z \leq 6$

25. The power for the above critical region and $p = 0.40$ for this test is:

(A) 0.9452 (B) 0.0548 (C) 0.8338 (D) 0.6331 (E) 0.1662

Questions 26 and 27 refer to the following exercise:

Let X and Y be two independent r.v. with corresponding distributions given by: $X \in N(m_X, \sigma_X^2 = 25)$ and $Y \in N(m_Y, \sigma_Y^2 = 36)$. In order to test the null hypothesis $H_0 : m_X = m_Y$ against the alternative hypothesis $H_1 : m_X \neq m_Y$, two r.s., each of size 30, have been taken from these populations, providing that: $\overline{x} = 82$ and $\overline{y} = 80$.

26. A 90% confidence interval for $(m_X - m_Y)$ is approximately given by:

(A) $(-0.34,\ 4.34)$ (B) $(-1.33,\ 5.33)$ (C) $(0.79,\ 4.79)$ (D) $(0.24,\ 4.34)$ (E) $(-0.79,\ 4.79)$

27. At the $\alpha = 10\%$ significance level, the decision of the test will be:

(A) Do not reject H_0 (B) - (C) Reject H_0 (D) - (E) -

Questions 28 to 30 refer to the following exercise:

In a given high school institution with 1200 registered students, we wish to estimate the proportion of students that plan to go to summer camps organized by this institution. We allow for a maximum error of ± 0.03 and assume a confidence level equal to 95%.

28. If we decide to take a random sample without replacement, the required sample size will be:

(A) 461 (B) 367 (C) More information is required (D) 1068 (E) 566

29. If we decide to take a random sample with replacement, the required sample size will be:

(A) 566 (B) 684 (C) 748 (D) 1068 (E) More information is required

30. If in the above question, the one for the random sample with replacement, we increase the confidence level, the required sample size will be:

(A) Larger (B) The same (C) More information is required (D) - (E) Smaller

EXERCISES (Time: 75 minutes)

A. (10 points, 25 minutes)

The following table includes information on the probability mass function a discrete r.v. X has under the null hypothesis ($P_0(x)$) and under the alternative hypothesis ($P_1(x)$).

X	1	2	3	4	5	6
$P_0(x)$	0	0.10	0.10	0.30	0.40	0.10
$P_1(x)$	0.30	0	0.25	0.05	0.10	0.30

A random sample of size $n = 1$ will be used to test the null hypothesis $H_0 : P(x) = P_0(x)$ against the alternative hypothesis $H_1 : P(x) = P_1(x)$.

i) Would you include the point $X = \{2\}$ in the critical region? Explain why or why not.

ii) Would you include the point $X = \{1\}$ in the critical region? Explain why or why not.

iii) At the 20% significance level and providing all relevant details used to obtain the required response, find the most powerful critical region for this test, and compute its probability of type II error. **Remark**: Before providing an answer to this item, take into account your responses to the previous items in this exercise.

B. (10 points, 25 minutes)

A random sample of 400 young individuals have been classified according to two variables: knowing one or more than one language and obtaining or not a job they were applying for, with the following results:

	Obtains the job	Does not obtain the job
Only one language	50	100
More than one language	200	50

We assume that all of the young individuals in the sample are homogeneous with respect to any other variable that may affect the study. With these data and at the 10% significance level, carry out the test of independence between the two variables under study: knowing one or more than one languages and obtaining or not a job they were applying for.

C. (10 points, 25 minutes)

Let X be a r.v. with probability mass function given by:

$$P_X(x;\theta) = e^{-2\theta} (2\theta)^x \left(\frac{1}{x!}\right), \quad \theta > 0, \quad x = 0, 1, \ldots$$

We wish to estimate the parameter θ and, in order to do so, we have taken a r.s. of size n, $X_1, X_2, \ldots, X_n$. It is known that the mean and the variance for the r.v. X are, respectively, $\mathrm{E}(X) = 2\theta$ and $\mathrm{Var}(X) = 2\theta$.

i) Obtain, **providing all relevant details**, the method of moments estimator of the parameter θ.

ii) Obtain, **providing all relevant details**, the maximum likelihood estimator of the parameter θ.

iii) Is the method of moments estimator of the parameter θ an unbiased estimator? Is it consistent? Is it efficient? **Remark**: You should appropriately justify all of your answers for this specific item. In addition, the Cramer-rao lower bound for a regular and unbiased estimator obtained from a r.s. is:

$$L_c = \frac{1}{n\mathrm{E}\left[\frac{\partial \ln P(X,\theta)}{\partial \theta}\right]^2}$$

SOLUTIONS TO MULTIPLE CHOICE QUESTIONS (exam type 0)

1: C	11: D	21: C
2: E	12: E	22: E
3: A	13: C	23: A
4: E	14: E	24: A
5: C	15: C	25: E
6: A	16: A	26: A
7: D	17: B	27: A
8: E	18: B	28: E
9: E	19: C	29: D
10: C	20: E	30: A

SOLUTIONS TO EXERCISES

Exercise A

We wish to test the null hypothesis that X is a r.v. with probability mass function $P_0(x)$ against the alternative hypothesis that its probability mass function is $P_1(x)$:

X	1	2	3	4	5	6
$P_0(x)$	0	0.10	0.10	0.30	0.40	0.10
$P_1(x)$	0.30	0	0.25	0.05	0.10	0.30

We have taken a r.s. of size$n = 1$; that is, we observe X.

i) Would you include the point $X \in \{2\}$ in the critical region?

Given that, under the probability mass function for the alternative hypothesis $P_1(x)$, this point has probability zero, the r.v. cannot take this value under the alternative hypothesis, but it can under the null hypothesis. Therefore, the point $X \in \{2\}$ is a point of no rejection for H_0 and, thus, it should **never** be included in the critical region for this test.

ii) Would you include the point $X \in \{1\}$ in the critical region?

Given that, under the probability mass function for the null hypothesis $P_0(x)$, this point has probability zero, the r.v. cannot take this value under the null hypothesis. Therefore, the point $X \in \{1\}$ is a rejection point for H_0 and, thus, it should **always** be included in the critical region for this test.

iii) At the $\alpha = 0.20$ significance level and **taking into account the responses we have provided to the previous items**, we have that the possible critical regions for this test are $\text{CR}_1 = \{1,3\}$, $\text{CR}_2 = \{1,6\}$ and $\text{CR}_3 = \{1,3,6\}$. This is so because the corresponding probabilities of type I error for these critical regions are:

$$P(X \in \text{CR}_1|P_0) = P(X = 1,3|P_0) = 0 + 0.10 = 0.10 \leq \alpha = 0.20$$

$$P(X \in \text{CR}_2|P_0) = P(X = 1,6|P_0) = 0 + 0.10 = 0.10 \leq \alpha = 0.20$$

$$P(X \in \text{CR}_3|P_0) = P(X = 1,3,6|P_0) = 0 + 0.10 + 0.10 = 0.20 \leq \alpha = 0.20$$

However, given that CR_3 includes the other two critical regions, we should only consider this CR, which will obviously be the most powerful one. In this case,

$$\text{Power} = P(X \in \text{CR}_3|P_1) = P(X = 1,3,6|P_1) = 0.30 + 0.25 + 0.30 = 0.85$$

$$\beta = P(\text{Type II Error}) = 1 - \text{Power} = 1 - 0.85 = 0.15$$

Exercise B

The information provided in this exercise is the following one:

	Obtains the job	Does not obtain the job	Total
Only one language	50	100	150
More than one language	200	50	250
Total	250	150	n=400

We should perform a test of independence. More specifically, we should test the null hypothesis H_0: knowing one or more than one language and obtaining or not a job they were applying for are independent variables, against the alternative hypothesis H_1: They are not independent variables. First of all, we should estimate the theoretical probabilities for each one of the classes for the two variables under study, whose independence we wish to test for. That is:

$$P(\text{job}) = 250/400 = 0.625, \quad P(\text{no job}) = 150/400 = 0.375$$

$$P(\text{one language}) = 150/400 = 0.375, \quad P(> \text{one language}) = 250/400 = 0.625$$

Under the null hypothesis of independence, we have that the joint probability will be equal to the product of the corresponding marginal probabilities. That is, $H_0 : p_{ij} = p_{i\bullet} \times p_{\bullet j}$. In this way, with the information provided in the sample and the estimated marginal probabilities above, we can build the table that would allow us to carry out the test.

Classes	n_{ij}	$\hat{p}_{ij}$	$n\hat{p}_{ij}$	$\frac{(n_{ij}-n\hat{p}_{ij})^2}{n\hat{p}_{ij}}$
(job, one language)	50	$0.625 \times 0.375 = 0.2344$	93.75	20.42
(no job, one language)	100	$0.375 \times 0.375 = 0.1406$	56.25	34.03
(job, > one language)	200	$0.625 \times 0.625 = 0.3906$	156.25	12.25
(no job, > one language)	50	$0.375 \times 0.625 = 0.2344$	93.75	20.42
	$n = 400$	1	$n = 400$	$z = 87.12$

We know that, under the null hypothesis, the test statistic Z converges to a chi square $\chi^2_{(I-1)(J-1)}$ distribution, where I is the number of rows in the table (i.e., $I = 2$) and J is the number of columns in the table (i.e., $J = 2$). At the approximate 10% significance level, the decision rule will be to reject the null hypothesis if

$$z > \chi^2_{1,0.10}$$

Given that

$$z = 87.12 > \chi^2_{1,0.10} = 2.71,$$

we reject the null hypothesis, so that there is no independence between the variables knowing one or more than one language and obtaining or not a job they were applying for. We should mention that, if we observe the information provided in the previous table and, more specifically, the corresponding values for the expected and observed probabilities therein, and the values for each of the components of the test statistic, it is possible that we only need to compute these values for one of the four classes in the table, which will result in a clear rejection on the null hypothesis.

Exercise C

The probability mass function for the random variable X is:

$$P_X(x;\theta) = e^{-2\theta}\,(2\theta)^x \left(\frac{1}{x!}\right), \quad \theta > 0, \quad x = 0, 1, \ldots$$

We wish to estimate the parameter θ and, in order to do so, a random sample of size n, $X_1, X_2, \ldots, X_n$, has been taken. It is known that the mean and the variance for the r.v. X are, respectively, $\mathrm{E}(X) = 2\theta$ and $\mathrm{Var}(X) = 2\theta$.

i) In order to be able to obtain the method of moments estimator of the parameter θ, we need to equate the first population moment $\alpha_1 = \mathrm{E}(X) = m$ to the first sample moment $a_1 = \overline{X}$. That is,

$$\alpha_1 = \mathrm{E}(X) = a_1 = \overline{X} \Longrightarrow 2\theta = \overline{X} \Longrightarrow \hat{\theta}_{\mathrm{MM}} = \frac{\overline{X}}{2}$$

ii) In order to be able to obtain the maximum likelihood estimator of the parameter θ, we have that likelihood function is given by:

$$L(\vec{x},\theta) = P(x_1,\theta)\ \cdots\ P(x_n,\theta) = e^{-2\theta}\,(2\theta)^{x_1}\left(\frac{1}{x_1!}\right)\cdots e^{-2\theta}\,(2\theta)^{x_n}\left(\frac{1}{x_n!}\right) = \frac{e^{-2n\theta}2^{\sum_{i=1}^n x_i}\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$$

If we take its natural logarithm, we have that:

$$\ln L(\vec{x},\theta) = -2n\theta + \left(\sum_{i=1}^n x_i\right)\ln(2) + \left(\sum_{i=1}^n x_i\right)\ln(\theta) - \ln\left[\prod_{i=1}^n x_i!\right]$$

If we take derivatives with respect to θ and make it equal to zero, we have that:

$$\frac{\partial \ln L(\vec{x},\theta)}{\partial\theta} = -2n + \frac{\sum_{i=1}^n x_i}{\theta} = 0$$

so that,

$$2n = \frac{\sum_{i=1}^n x_i}{\theta} \Longrightarrow \hat{\theta}_{\text{ML}} = \frac{\sum_{i=1}^n X_i}{2n} = \frac{\overline{X}}{2}$$

iii) Unbiasedness: In order to verify if the method of moments estimator is an unbiased estimator of θ, we compute its expected value.

$$\mathrm{E}\left(\hat{\theta}_{\text{MM}}\right) = \mathrm{E}\left(\frac{\overline{X}}{2}\right) = \frac{1}{2}\mathrm{E}\left(\overline{X}\right) = \frac{1}{2}\mathrm{E}\left(X\right) = \left(\frac{2\theta}{2}\right) = \theta$$

Therefore, it is an unbiased estimator of θ.

Consistency: We compute the variance of the method of moments estimator of θ.

$$\mathrm{Var}\left(\hat{\theta}_{\text{MM}}\right) = \mathrm{Var}\left(\frac{\overline{X}}{2}\right) = \frac{1}{4}\mathrm{Var}\left(\overline{X}\right) = \left(\frac{1}{4}\right)\left[\frac{\mathrm{Var}(X)}{n}\right] = \frac{2\theta}{4n} = \frac{\theta}{2n}$$

Given that it is an unbiased estimator whose variance tends zero as n goes to infinity, we can state that the method of moments estimator of θ is a consistent estimator.

Efficiency: In order to be able to verify if the method of moments estimator of θ is efficient, we compute the Cramer-Rao lower bound for regular and unbiased estimators:

$$L_c = \frac{1}{n\mathrm{E}\left(\frac{\partial \ln P(x;\theta)}{\partial\theta}\right)^2}$$

$$\ln P(x;\theta) = -2\theta + x\ln(2\theta) - \ln(x!)$$

$$\frac{\partial \ln P(x;\theta)}{\partial\theta} = -2 + \frac{x}{\theta} = \frac{x-2\theta}{\theta}$$

$$\mathrm{E}\left[\frac{X-2\theta}{\theta}\right]^2 = \frac{1}{\theta^2}\mathrm{E}\left[X-2\theta\right]^2 = \frac{1}{\theta^2}\mathrm{Var}(X) = \frac{1}{\theta^2}(2\theta) = \frac{2}{\theta}$$

If we now replace this expected value in the Cramer-Rao lower bound formula, we have that:

$$L_c = \frac{1}{n\left(\frac{2}{\theta}\right)} = \frac{\theta}{2n} = \mathrm{Var}(\hat{\theta}_{\text{MM}})$$

Therefore, the method of moments estimator is an efficient estimator of θ.