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Abstract

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Keywords

Stochastic programming – Scenario tree – Complete recourse – Deterministic equivalent model

Mathematics Subject Classification (2000) 60H35 – 65C20 – 68Q25

Footnotes

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ORIGINAL PAPER

The value of the stochastic solution in multistage problems

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Abstract We generalize the definition of the bounds for the optimal value of the objective function for various deterministic equivalent models in multistage stochastic programs. The parameters *EVPI* and *VSS* were introduced for two-stage models. The parameter *EVPI*, *the expected value of perfect information*, measures how much it is reasonable to pay to obtain perfect information about the future. The parameter *VSS*, *the value of the stochastic solution*, allows us to obtain the goodness of the expected solution value when the expected values are replaced by the random values for the input variables. We extend the definition of these parameters to the multistage stochastic model and prove a similar chain of inequalities with the lower and upper bounds depending substantially on the structure of the problem.

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Keywords Stochastic programming · Scenario tree · Complete recourse · Deterministic equivalent model

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1 Introduction

Stochastic programs in general and multistage programs in particular have the reputation of being computationally difficult to solve. In many cases, even before solving the stochastic model, the user must investigate and solve several deterministic models, each corresponding to one particular scenario or to a group of them. Then, the optimal solution of these programs must be combined to see if the scenarios, stages and other aspects of the model are well defined, and the solution of the stochastic model (*RP*, recursion problem) is really justified, or if, on the contrary, another selection is needed. There are two concepts, namely *the expected value of perfect information (EVPI)* and *the value of the stochastic solution* (*VSS*), see Birge and Louveaux (1997), that can help to do this. Both were developed for the case of two-stage problems, see Madansky (1960), Manne (1974), Chao (1981), Birge (1982, 1988) and Louveaux and Smeers (1988), among others, and can be directly calculated.

For the maximization models in particular, the following inequalities are satisfied, see Madansky (1960):

$$EEV \leq RP \leq WS,$$

where the value *EEV* denotes the expected result of using the solution of the deterministic model *EV*, the one obtained by replacing all random variables by their expected values; the *WS* value, known in the relevant literature as the *wait and see* solution value, denotes the expected value of using the optimal solution for each scenario (*WS* models); and the *RP* value, also known as the *here and now* solution, denotes the optimal solution value to the recursion problem.

In this context, the difference $EVPI = WS - RP$ denotes the *expected value of perfect information* and compares *here-and-now* and *wait-and-see* approaches. A small *EVPI* indicates a low additional profit when we reach perfect information. $VSS = RP - EEV$ denotes *the value of the stochastic solution* and compares the *here-and-now* and expected values approaches. A small *VSS* means that the approximation of the stochastic program by the program with expected values instead of random variables is a good one.

In two-stage models, obtaining *VSS* means calculating *EEV*. This can be obtained as follows: (1) solve the related average scenario problem, *EV*; (2) fix the first stage solution for each scenario (*WS* model), at the optimal one obtained for the first stage of the *EV* problem; (3) solve the resulting problem for each scenario, and (4) calculate the expectation over the set of scenarios of the value at the objective function of these modified *WS* problems.

As mentioned above, the parameters have been studied in the open literature for the two-stage case. In this work, we generalize those parameters to the multistage case. This generalization entails considering various issues. In particular, it is not clear which variables must be fixed in the *WS* models, see Valente (2002). A trivial solution to this problem would be to fix only the first stage solutions (decisions), as in the two-stage case, so that in the solution of the corresponding *WS* models all the variables of the following stages are free to be adapted to the performance of the different scenarios. This procedure, nevertheless, can become a paradox in some cases, since it can happen that the first stage solution in the *EV* problem performs better than the solution of the *RP* one! The reason is that in the multistage case, the *RP* model contains *nonanticipativity* constraints in later stages, which are ignored (relaxed) when the *WS* models are solved. Note that the *WS* models denote the model for each scenario and they are completely independent.

Therefore, before the family of independent *WS* models are solved, we must solve a chain of models (starting with the *RP* model). The decision variables of the first stage are fixed to the solution of the *EV* problem, in order to compare the *RP* solution consistently with the *EV* solution.

There are some studies in which the above mentioned expected values appear to check the quality of the stochastic solution in multistage models, for example, see Alonso et al. (2004). Nevertheless, in contrast to the two-stage case and as far as we know, there are no studies where mathematical properties and formal relations are derived.

The rest of the paper is organized as follows. Section 2 presents the Deterministic Equivalent Model of the multistage optimization model to be studied. Section 3 presents the solution to the average scenario and its expected value in each time period, and proves some properties between the related chain of values. Section 4 deals with the dynamic solution of the average scenario model and also includes some mathematical results about the related expected values. Section 5 reports all the expected values for the multistage optimization model of the well-known investor problem. Section 6 concludes. Some computational insight to compute the expected values defined efficiently is included as an Appendix.

2 Deterministic equivalent model, *DEM*

A general mixed 0–1 multistage optimization model has the following form:

$$\begin{aligned} &\max \sum_{t\in\mathcal{T}} c_t z_t \\ &\text{s.t.} \quad C_{1t}z_1 + C_{2t}z_2 + \cdots + C_{tt}z_t = d_t \qquad \forall t \in \mathcal{T}, \\ &\qquad\ z_t \in Z \subseteq \mathbb{R}^n \qquad \forall t \in \mathcal{T}, \end{aligned} \tag{1}$$

where c_t is the vector of the objective function coefficients, C_{it} gives the constraint matrix for the pair of stages i and t, $i = 1, \ldots, t$, and d_t is the right-hand-side vector (*rhs*) for stage t, $t \in \mathcal{S} \subset \mathcal{T}$, $\mathcal{S}$ is the set of stages (time periods in which a decision is taken) and $\mathcal{T}$ is the set of time periods in a time horizon to be considered; z_t is the (n-dimensional) vector of variables for stage t; and Z is a non-empty closed set that may include an integer or 0–1 character for any of the components of vector z. This last situation and the loss of convexity of the feasible region that it produces makes the theoretical treatment of the problem more difficult.

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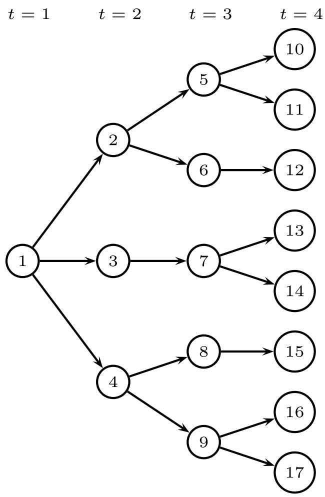

$\Omega = \Omega_1 = \{10, 11, \ldots, 17\}$; $\Omega_2 = \{10, 11, 12\}$

$\mathcal{G}_2 = \{2, 3, 4\}$; $t(5) = 3$

$\mathcal{G} = \{1, 2, 3, ..., 17\}$

$\mathcal{S} = \{1, 2, 3\}, S = 3, \mathcal{T} = \{1, 2, 3, 4\}, T = 4$

$\pi(9) = 4, \pi(14) = 7$

Fig. 1 Scenario tree

In particular, in the present study, we consider a multistage model in which the constraints only relate two consecutive periods. Thus, the model can be expressed as

$$
\begin{aligned}
&\max \sum_{t \in \mathcal{T}} c_t z_t \\
&\text{s.t.} \quad C'_t z_{t-1} + C_t z_t = d_t \qquad \forall t \in \mathcal{T}, \\
&\qquad\ \ z_t \in Z \qquad\qquad\qquad\quad\ \ \forall t \in \mathcal{T},
\end{aligned}
\tag{2}
$$

where $C_t = C_{tt}$ and $C'_t = C_{t-1,t}$. Moreover, in many real cases, the model must be extended to consider uncertainty in some of the main parameters, in our case, the objective function, the rhs coefficients, and the constraint matrices.

The uncertainty in the stochastic parameters is dealt with via a *scenario tree* based approach. It is a structure representing the evolution of information over the stages, see Fig. 1. Let us define a symmetric and balanced scenario tree as a tree where all the scenarios have the same probability and its form is symmetric. In a scenario tree, two scenarios that share a common history until stage t are indistinguishable until that stage, and thereafter they are represented by distinct paths. Thus each distinct scenario represents a path from the root node to a leaf node of the scenario tree. In the absence of appropriate approximations these trees can become extremely large, making the model difficult to manage and solve.

Each node in the tree can also be associated with a scenario group such that two scenarios belong to the same group in a given stage provided that they have the same realizations of the uncertain parameters up to that stage. In accordance with the *nonanticipativity* principle, see Rockafellar and Wets (1991), both scenarios should have the same value for the related variables with the time index up to the given stage.

The following additional notation is used in the paper:

Ω, the set of scenarios that represent the stochasticity of the uncertain parameters.
$\mathcal{G}$, the set of scenario groups. Note: the groups are numbered consecutively.

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$\mathcal{G}_t$, the set of scenario groups in the period t, for $t \in \mathcal{T}$ ($\mathcal{G}_t \subset \mathcal{G}$). Note: $|\mathcal{G}_1| = 1$ and $1 \in \mathcal{G}_1$.
Ω_g, the set of scenarios that belong to the group g, for $g \in G$ ($\Omega_g \subset \Omega$).
$t(g)$, the stage of scenario group g such that $g \in \mathcal{G}_{t(g)}$.
$\pi(g)$, the immediate ancestor of node g.

Let us assume that the uncertainty can be modeled by a finite number of scenarios. Let ξ be the stochastic variable whose realizations correspond to the different scenarios. Hence $\xi(\omega) = (c(\omega), C(\omega), C'(\omega), d(\omega))$, $\omega \in \Omega$.

Let z_t^ω denote the vector of the variables related to stage t under scenario ω, for $t \in \mathcal{T}$ and $\omega \in \Omega$. Thus, for each time period $t \in \mathcal{T}$, the z_t^ω variable is fixed optimally before the future realizations $(\xi_{t+1}, \ldots, \xi_T)$ are observed. Once ξ_t is observed, the decisions for the stage t, say z_t, are taken. The *nonanticipativity* principle can be expressed as the set

$$\left\{\left(z_t^\omega\right) : z_t^\omega = z_t^{\omega'}, \forall \omega, \omega' \in \Omega_g, \omega \neq \omega', g \in \mathcal{G}_t, t \in \mathcal{S}\right\}.$$

The *splitting variable* representation (Alonso et al. 2003) of the mixed 0–1 *DEM* of the stochastic version with complete recourse of the deterministic multistage problem (2) can thus be expressed as follows:

$$\begin{aligned}
RP = \max & \sum_{\omega \in \Omega} \sum_{t \in \mathcal{T}} w^\omega c_t^\omega z_t^\omega \\
\text{s.t.} \quad & C_t^{'\omega} z_{t-1}^\omega + C_t^\omega z_t^\omega = d_t^\omega \qquad \forall \omega \in \Omega, t \in \mathcal{T}, \\
& z_t^\omega - z_t^{\omega'} = 0 \qquad \forall \omega, \omega' \in \Omega_g, \omega \neq \omega', g \in \mathcal{G}_t, t \in \mathcal{S}, \\
& z_t^\omega \in Z \qquad \forall \omega \in \Omega, t \in \mathcal{T},
\end{aligned} \tag{3}$$

where w^ω is the likelihood assigned to scenario ω, c_t^ω is the row vector of the objective function coefficients, C_t^ω and $C_t^{'\omega}$ are the constraint matrices, and d_t^ω is the *rhs* vector, for $\omega \in \Omega, t \in \mathcal{T}$. Note that $C_t^{'\omega} = C_t^{'\omega'}$, $C_t^\omega = C_t^{\omega'}$, $c_t^\omega = c_t^{\omega'}$ and $d_t^\omega = d_t^{\omega'}$ for $\omega, \omega' \in \Omega_g, \omega \neq \omega', g \in \mathcal{G}_t, t \in \mathcal{S}$.

The solution of this model, z_t^ω, takes into account the different fluctuations from the unknown elements of the problem. All the decisions are adjusted in time when new information is obtained, except for the decisions corresponding to the first time period.

Sometimes it is necessary to carry out the following partition for the variables corresponding to the stage t, z_t: the state or decision variables corresponding to the stage, denoted by x_t, and the recourse variables, denoted by y_t. These last ones (which in general are continuous variables) correspond to recourse actions taken at stage t in order to correct or to offset the strategic decisions taken up to stage t. Thus, $z_t \equiv (x_t, y_t)$, $c_t = (a_t, b_t)$, $C_t = (A_t, B_t)$ and $C'_t = (A'_t, B'_t)$. In general, the state and recourse variables are related and defined by the balance constraints

$$A_t^{'\omega} x_{t-1} + A_t^\omega x_t + B_t^{'\omega} y_{t-1} + B_t^\omega y_t = d_t^\omega.$$

Let *RP* value be the optimal value of the objective function in the *RP* model (3).

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3 On the solution of the average scenario deterministic model, EV, and its expected value in t, EEV_t

Let Z_{EV} be the optimal value of the objective function in the average scenario deterministic model, EV, which can be expressed as follows:

$$\begin{aligned} Z_{EV} = \max & \sum_{t\in\mathcal{T}} \overline{a}_t x_t + \overline{b}_t y_t \\ \text{s.t.} \quad & \overline{A}'_t x_{t-1} + \overline{A}_t x_t + \overline{B}'_t y_{t-1} + \overline{B}_t y_t = \overline{d}_t \quad \forall t \in \mathcal{T}, \\ & x_t \in X,\ y_t \in Y \quad \forall t \in \mathcal{T}, \end{aligned} \tag{4}$$

where $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are non-empty closed sets that can include integer conditions for some of the components of x and y, n and m are the dimensions of $x \in X$ and $y \in Y$, respectively, $\overline{a}_t$, $\overline{b}_t$, $\overline{A}'_t$, $\overline{A}_t$, $\overline{B}'_t$, $\overline{B}_t$ and $\overline{d}_t$ denote the expected values of the vectors and matrices a_t^ω, b_t^ω, $A_t'^\omega$, A_t^ω, $B_t'^\omega$, B_t^ω and d_t^ω over the set of scenarios ω, respectively, for $t \in \mathcal{T}$. Let $(\overline{x}_t, \overline{y}_t)$ be the optimal solution of the model (4) for $t \in \mathcal{T}$.

The quality of the approach provided by the average scenario deterministic model, EV, can be measured by the value of VSS.

Definition 1 Let the *expected result in t of using the expected value solution*, denoted by EEV_t for $t = 2, \dots, T$, be the optimal value of the RP model (3), where the decision variables until stage $t-1$, $(x_1, \dots, x_{t-1})$, are fixed at the optimal values obtained in the solution of the average scenario model (4). That is,

$$EEV_t = \begin{cases} RP \text{ model } (3) \\ \text{s.t.} \quad x_1^\omega = \overline{x}_1 & \forall \omega \in \Omega, \\ \quad\quad \dots \\ \quad\quad x_{t-1}^\omega = \overline{x}_{t-1} & \forall \omega \in \Omega. \end{cases}$$

If we extend the definition to $t = 1$ and define $EEV_1 = RP$, we get the sequence of expected values

$$EEV_1, EEV_2, \dots, EEV_T,$$

for which the following relation holds.

Proposition 1 *For any multistage stochastic program*

$$EEV_{t+1} \le EEV_t \quad \forall t = 1, \dots, T-1.$$

Proof Any feasible solution of the EEV_{t+1} problem is also a solution of the EEV_t problem, because the feasible region of the first problem has one set of constraints more than the second problem (the set where the decision variables of stage t are fixed). Consequently, the proof is trivial. If the EEV_{t+1} model is infeasible for any stage t, i.e. $EEV_{t+1} = -\infty$, the inequality is satisfied trivially. Finally, if the EEV_{t+1} model is unbounded for any t, then the EEV_t model is unbounded too, and therefore $EEV_{t+1} = EEV_t = \infty, \forall t$. □

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Definition 2 We define the *value of the stochastic solution in* t, denoted by VSS_t, as

$$VSS_t = RP - EEV_t \quad \forall t = 1, \ldots, T.$$

Corollary 1 *For any multistage stochastic program*

$$0 \leq VSS_t \leq VSS_{t+1} \quad \forall t = 1, \ldots, T-1.$$

Proof The inequalities can be deduced from Proposition 1. □

This sequence of non-negative values represents the cost of ignoring uncertainty until stage t in the decision making of multistage models.

Proposition 2 *For multistage stochastic linear programs with deterministic constraint matrices and deterministic objective coefficients, the following inequalities are satisfied*:

$$VSS_t \leq EV - EEV_t \quad \forall t = 1, \ldots, T.$$

Proof The inequality $WS \leq EV$ is valid for stochastic programs where uncertainty is only in the *rhs*, because in this case the Jensen inequality can be applied. Since $RP \leq WS$, the above inequality holds. □

In particular, the expected value of the solution that provides the average scenario problem, *EEV*, defined for two-stage models, is equal to the value of EEV_T in multistage models. This value enables us to calculate the maximum cost that we would be prepared to pay to ignore uncertainty over the time horizon.

The intermediate values, EEV_t, and then VSS_t, for $t = 2, \ldots, T$, give us information about a suitable choice of the number of stages in the model. Similar successive values $EEV_t \approx EEV_{t+1}$, and then $VSS_t \approx VSS_{t+1}$, would indicate that the deterministic problem until stage t is a good approach to the stochastic one, and therefore it would not be necessary to define later stages.

A sufficient condition for $EEV_t = EEV_{t+1}$, and then $VSS_t = VSS_{t+1}$, is the independence of $z_{t+1}(\omega)$ of the scenario ω. This means that the optimal values at stage $t+1$ are insensitive to the value of the random elements. In such situations, finding the optimal solution for one particular $\xi^{(\omega)}$ ($\xi^{(\overline{\omega})}$, for example) would yield the same result, and it is unnecessary to define a further stage.

In models where many variables have been fixed, there is often no feasible solution. Thus, if we find that all the problems $EEV_2, EEV_3, \ldots, EEV_t, \ldots$ are infeasible, this will not give us too much information about the quality of the approach obtained. An alternative that sometimes avoids this problem consists of not fixing those variables that generate infeasibility in the model. Thus we get a more flexible definition of the model that provides an estimation of EEV_t. This definition is as follows.

Definition 3 Let the *feasible expected value in t of using the solution of the average scenario model*, denoted by $E\hat{E}V_t$, be the optimal value of the *RP* model (3), where

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the decision variables until stage $t-1$ are fixed at zero if they are fixed at zero in the optimal solution of the average scenario model (4). That is,

$$E\hat{E}V_t = \begin{cases} RP \text{ model (3)} \\ \text{s.t.} \quad x_1^{\omega} \le \overline{x}_1 M_1 & \forall \omega \in \Omega, \\ \quad\quad \dots \\ \quad\quad x_{t-1}^{\omega} \le \overline{x}_{t-1} M_{t-1} & \forall \omega \in \Omega, \end{cases}$$

where $M_1, \dots, M_{t-1}$ are sufficiently large constants.

Proposition 3 *For any multistage stochastic program it holds that*

$$EEV_t \le E\hat{E}V_t \quad \forall t = 1, \dots, T.$$

Proof Trivial. □

4 Dynamic solution of the average scenario, EV_g

A more realistic use of the expected value solution in multistage problems can be obtained by solving a model, say EV_g, for each scenario group at each stage of the problem. In this way the estimates are updated and, although they are based on average values, they add more precise information to the model. In this case a deterministic solution is also obtained, but it is not an anticipative one.

The expected result of using these *dynamic solutions* of the model based on average scenarios is obtained immediately from the solution of these models.

Next we describe the calculation procedure.

Step 1: $t = 1$. Obtain the optimal value, Z_{EV}^1, of the corresponding average scenario model, EV_1 defined for $g = 1 \in \mathcal{G}_1$, and save the optimal values of the first stage variables (x_1, y_1), say $(\overline{x}_1, \overline{y}_1)$.
Step 2: $t := t + 1$. Define the set of $|\mathcal{G}_t|$ average scenario problems, EV_g, for the scenario subtrees corresponding to each group of scenarios at the next stage $g \in \mathcal{G}_t$, where the random parameters of subsequent stages are estimated by their expected values, and all the variables of the previous stages are fixed at the optimal solution values obtained in the chain $EV_{\pi(g)}$, $g \in \mathcal{G}_\tau$, $\tau = 1, \dots, t-1$.
After solving the $|\mathcal{G}_t|$ problems, the optimal values, Z_{EV}^g, and the optimal values of the variables of the current stage $(x_t, y_t)^g$, say $(\overline{x}_t, \overline{y}_t)^g$, are saved, for $g \in \mathcal{G}_t$.
Step 3: If $t < T$, go to Step 2.
Step 4: Define the set of $|\mathcal{G}_T| = |\Omega|$ average scenario problems, EV_g, $g \in \mathcal{G}_T$, for the scenario subtrees corresponding to each group of scenarios at the last stage, where all the variables of the previous stages are fixed, except for the last one, at the optimal solution values obtained in the chain of models $EV_{\pi(g)}$, $g \in \mathcal{G}_\tau$, $\tau = 1, \dots, t-1$.

Definition 4 We define the *expected result in t of using the dynamic solution of the average scenario*, and we denote it by $EDEV_t$, for $t = 1, \dots, T$, as the expected value of the optimal values of the EV_g problems, with $g \in G_t$, that is

$$EDEV_t = \sum_{g \in \mathcal{G}_t} w^g Z_{EV}^g, \quad t = 1, \dots, T,$$

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where Z_{EV}^{g} is the optimal value for the model EV_g and w^g represents the likelihood of the scenario group g, obtained as $w^g = \sum_{\omega \in \Omega_g} w^\omega$.

The proof that $EDEV_1 = EV$, for $t = 1$, is obvious. Moreover, we can observe that $EDEV_T = EEV_T$, whenever the scenario tree is a symmetric and balanced one.

Proposition 4 *For multistage stochastic linear programs with deterministic constraint matrices and deterministic objective coefficients,*

$$EDEV_{t+1} \leq EDEV_t, \quad \forall t = 1, \ldots, T-1.$$

Proof The proposition follows by application of Jensen's inequality in each time period t. □

Definition 5 Let us define the *dynamic value of the stochastic solution*, say VSS^D, as

$$VSS^D = RP - EDEV_T.$$

Proposition 5 *For any multistage stochastic program,*

$$VSS^D \geq 0.$$

Proof The optimal solution of $EDEV_T$ is a feasible solution for the stochastic model, RP, so $EDEV_T \leq RP$. If $EDEV_T$ has no feasible solution, the result is trivial. □

Computationally, to obtain this value we must solve a large number of submodels, in particular, $|\mathcal{G}_1| + |\mathcal{G}_2| + \cdots + |\mathcal{G}_T| = |\mathcal{G}|$, but they have small dimensions.

Proposition 6 *For any multistage stochastic program,*

$$EDEV_{t+1} \leq EDEV_t \quad \forall t = S, \ldots, T-1.$$

Proof The optimal solution of $EDEV_{t+1}$ is a feasible solution of $EDEV_t$ in each time period t of the last stage. □

Definition 6 Let us define the *dynamic value of the stochastic solution* for each time period t in the last stage, say VSS_t^D, as

$$VSS_t^D = RP - EDEV_t \quad \forall t = S, \ldots, T.$$

Proposition 7 *For any multistage stochastic program*

$$0 \leq VSS_t^D \leq VSS_{t+1}^D \quad \forall t = S, \ldots, T-1.$$

Proof The first inequality is satisfied, since the optimal solution of $EDEV_t$ is a feasible solution of the stochastic model, RP, so $EDEV_t \leq RP$, $\forall t = S, \ldots, T$. If $EDEV_t$ has no solution for any t, the result is trivial. The second inequality follows directly from Proposition 6. □

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In the example described in Sect. 5, a particular case of symmetric and balanced scenario tree, we can see that for the last time period the expected value, $EDEV_T$, coincides with the expected result of using the expected value solution, EEV_T. In general, a sufficient condition for $EEV_T = EDEV_T$ is the independence of the optimal solution $(\overline{x}_t, \overline{y}_t)^g$ of the scenario group g, $g \in \mathcal{G}_t$. This means that the optimal solutions are insensitive to the scenario group in which they have been obtained, and it does not matter in which of them we solve the deterministic model of the average scenario.

In other types of trees, the dynamic solution of the average scenario makes its use more realistic, given that it provides a nonanticipative solution. Moreover, as it is observed in the example described below, the deterministic dynamic model of the average scenario approaches the stochastic model better than the *classical* deterministic model of the average scenario.

5 Case study. The investor's problem

Let us consider the famous problem, see Birge and Louveaux (1997), whose objective consists of obtaining an investment policy which maximizes the utility of the returns at the end of the time horizon.

The investor wishes to invest d_1 in some of, say I, considered investments. Let the tuition goal be d_T; exceeding this amount at the end of the time horizon provides an additional income of $q^+\%$ of the excess, while not meeting the goal would result in borrowing at the rate $q^-\%$, $q^- > q^+$. The investor plans to revise his investment at each time period using additional information that will gradually become available in the future. The time periods are indexed by $t = 1$ for the initial decision, by $t = 2, \dots, T-1 = S$ for the revisions and by $t = T$ for the time horizon. The main uncertainty is the return r_i^t on each investment i at the end of time period t. The investment decisions, say x_i^t, denote the volume of investment in the asset i at the beginning of time period t. The state decisions, say y^+ and y^-, denote the surplus and deficit, respectively, at the end of the last period, T.

In order to illustrate the different bounds and expected value solutions, we will take the following values for the parameters: $|I| = 2$ ($i = 1$, stocks, and $i = 2$, bonds); $S = T - 1 = 3$ stages or revision periods and $T = 4$ for number of periods in the time horizon.

The initial capital budget is $d_1 = 55$ thousand euros, and after T years we will have a wealth that we would like to exceed a tuition goal of $d_T = 80$ thousand euros. We assume an income of $q^+\% = 1\%$ over the excess, while not meeting the goal would lead to borrowing at cost $q^-\% = 4\%$. The average unit returns are $r_1^t = 1.155$ for stocks and $r_2^t = 1.13$ for bonds, for $t \in T$. We assume that eight possible scenarios may occur over the three decision periods. The scenarios correspond to independent and equal likelihoods of having (inflation-adjusted) returns of either 1.25 for stocks and 1.14 for bonds or 1.06 for stocks and 1.12 for bonds over each of the three revision periods. With the scenarios defined here, we assign probabilities for each ω, $w^\omega = 0.125$. So, the deterministic equivalent model to solve in compact version is as

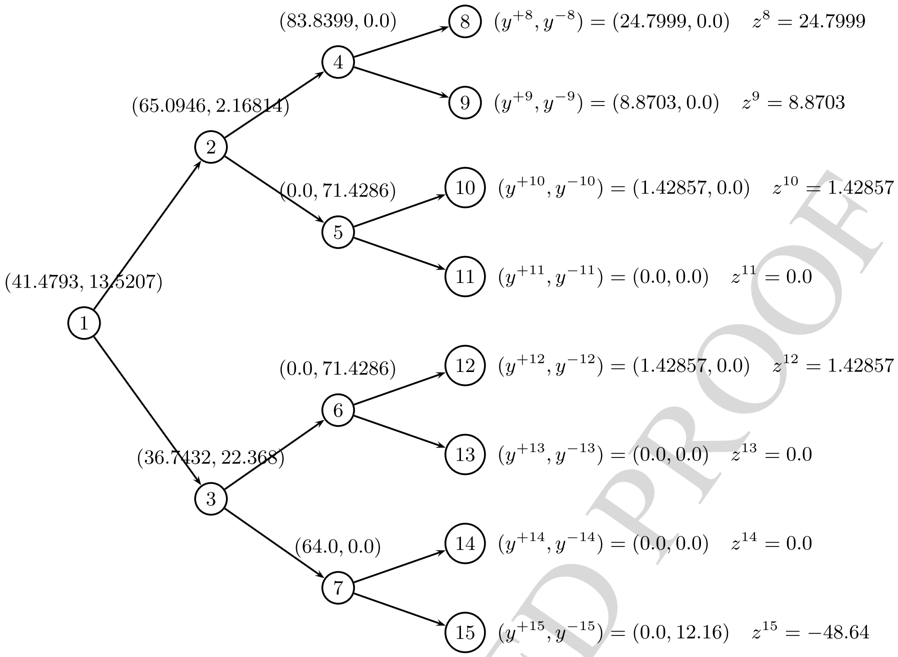

Fig. 2 Optimal stochastic solution, *RP*

follows:

$$
\begin{aligned}
RP = \max & \sum_{g\in\mathcal{G}_T} w^g\left(q^+ y^{+g} - q^- y^{-g}\right) && \\
\text{s.t.} \quad & \sum_{i\in\mathcal{I}} x_i^1 = d_1, && \\
& -\sum_{i\in\mathcal{I}} r_i^{\pi(g)} x_i^{\pi(g)} + \sum_{i\in\mathcal{I}} x_i^g = 0 && \forall g \in \mathcal{G}_t, t = 2, \ldots, S, \\
& \sum_{i\in\mathcal{I}} r_i^{\pi(g)} x_i^{\pi(g)} - y^{+g} + y^{-g} = d_T && \forall g \in \mathcal{G}_T, \\
& x_i^g \geq 0 && \forall i \in \mathcal{I}, g \in \mathcal{G}, \\
& y^{+g}, y^{-g} \geq 0 && \forall g \in \mathcal{G}_T.
\end{aligned}
\tag{5}
$$

Notice that problem (5) is always feasible, due to the assumed unlimited possibilities of borrowing. Figure 2 shows the investment plan.

It can be observed a natural diversification of the investment. The solution tells us, for example, to invest 75.42% in stocks and the rest in bonds at the first stage. At the second stage, if high returns are observed, the intent is to invest 96.77% in stocks, while if low returns are observed, it does not risk so much and proposes to invest 62.16% in stocks. The investment policy is such that in 87.5% of the cases a non-negative utility is reached, i.e. the target is reached in all cases except for the most unfavorable one. If we implement this policy in each time period, we would have an expected utility of $EEV_1 = RP = -1.51408$ thousand euros, which is negative and represents an expected loss over the tuition goal.

The multiperiod formulation of the deterministic model of the average scenario is as follows:

$$
\begin{aligned}
Z_{EV} &= \max q^+ y^+ - q^- y^- \\
\text{s.t.}\quad & \sum_{i\in\mathcal{I}} x_i^1 = d_1 \\
& -\sum_{i\in\mathcal{I}} \overline{r}_{it-1} x_i^{t-1} + \sum_{i\in\mathcal{I}} x_i^t = 0 \qquad \forall t = 2,\dots,S, \\
& \sum_{i\in\mathcal{I}} \overline{r}_{iT} x_i^T - y^+ + y^- = d_T, \\
& x_i^t, y^+, y^- \ge 0 \qquad\qquad \forall i \in \mathcal{I}, t \in T,
\end{aligned}
\tag{6}
$$

where $\overline{r}_{it}$ represents the average return on asset i valid throughout time period t. The optimal objective function value of (6) is $Z_{EV} = 4.7394$. The investment policy that proposes the optimal solution of (6) is shown in the following scheme:

$$
\begin{array}{ccc}
t = 1 & t = 2 & t = 3 \\
(x_1^1, x_2^1) = (55, 0) & (x_1^2, x_2^2) = (63.525, 0) & (x_1^3, x_2^3) = (73.3714, 0).
\end{array}
\tag{7}
$$

Figure 3 shows the implementation of this policy at the first stage.

The implementation of this solution, ignoring the random returns for the first time period, allows us to calculate the expected result in $t = 2$ using the expected value solution, say $EEV_2 = -1.9631$ thousand euros.

We can measure the expected result in $t = 3$ using the expected value solution over the two first stages, that is EEV_3, which is infeasible. This is because the optimal solution of the average scenario model tells us to invest 63.255 thousand euros in stocks at the second stage, a value that is not available if there is the total investment in stocks in the first stage and the returns go down. That optimal selection advises us to invest everything in stocks and nothing in bonds in each of the time periods. If we implement this selection, by fixing the decision variables corresponding to the investment in bonds for the second stage at zero, we obtain the estimation of the expected result in $t = 3$ using the expected value solution over the first two stages, say $E\hat{E}V_3 = -2.29698$ thousand euros. The policy of investments is shown in Fig. 4.

Finally, the feasible implementation of the asset selection provided by the average scenario model appears in the tree in Fig. 5. The expected result of implementing this selection over all three time periods is $E\hat{E}V_4 = -3.78792$, which coincides with the general value of the expected result using the expected value solution, say EEV, introduced by Birge (1982) for two-stage problems. It represents the greatest expected loss, even when the average scenario promises the greatest benefit.

In Sect. 4, we proposed to make use of the solution based on the dynamic average scenario model that provides a chain of non-anticipative decisions. The solution of the deterministic average models EV_g, for each scenario group $g \in \mathcal{G}$, is given in the following table:

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		$Z_{EV}^4 = 19.2578$ $(x_1, x_2)^4 = (85.9375, 0)$	$Z_{EV}^8 = 27.4219$ $(x_1, x_2)^8 = (27.4219, 0)$
$Z_{EV}^1 = 4.74$	$Z_{EV}^2 = 11.71$ $(x_1, x_2)^2 = (68.75, 0)$	$Z_{EV}^5 = 4.17063$ $(x_1, x_2)^5 = (72.875, 0)$	$Z_{EV}^9 = Z_{EV}^{10} = Z_{EV}^{12} = 11.0938$ $(x_1, x_2)^g = (11.0938, 0)$, $g = 9, 10, 12$
$(x_1, x_2)^1 = (55, 0)$	$Z_{EV}^3 = -8.90$ $(x_1, x_2)^3 = (58.3, 0)$	$Z_{EV}^6 = 4.17063$ $(x_1, x_2)^6 = (72.875, 0)$	$Z_{EV}^{11} = Z_{EV}^{13} = Z_{EV}^{14} = -11.01$ $(x_1, x_2)^g = (0, 2.7525)$, $g = 11, 13, 14$
		$Z_{EV}^7 = 19.2578$ $(x_1, x_2)^7 = (61.798, 0)$	$Z_{EV}^{15} = -57.9765$ $(x_1, x_2)^{15} = (0, 14.4941)$

By fixing the values of the first stage variables at the optimal values in the solution for $g = 1$, we construct and solve two new deterministic models, say EV_2 and EV_3, associated with the two scenario groups in stage 2. The coefficients which define the model EV_2 are those corresponding to nodes 1 and 2, until stage 2 of the scenario tree, and the expected values between nodes 4 and 5 for the third stage. We construct and solve the models corresponding to the scenario groups of the third stage obtaining for EV_4 an investment for the third period given by $(x_1, x_2)^4 = (85.9375, 0)$. For EV_5 and EV_6, we obtain the same investment plan for the third period. Finally, for EV_7, we obtain some other investment plan for the third period. We now go on to the scenario groups in the fourth stage, which are the scenarios for each of which we must solve a problem and obtain the investment plan for the fourth period.

Some conclusions can be drawn from this chain of values. If we show the decisions in a tree, it results to be the same tree as in Fig. 5, i.e. the one corresponding to the

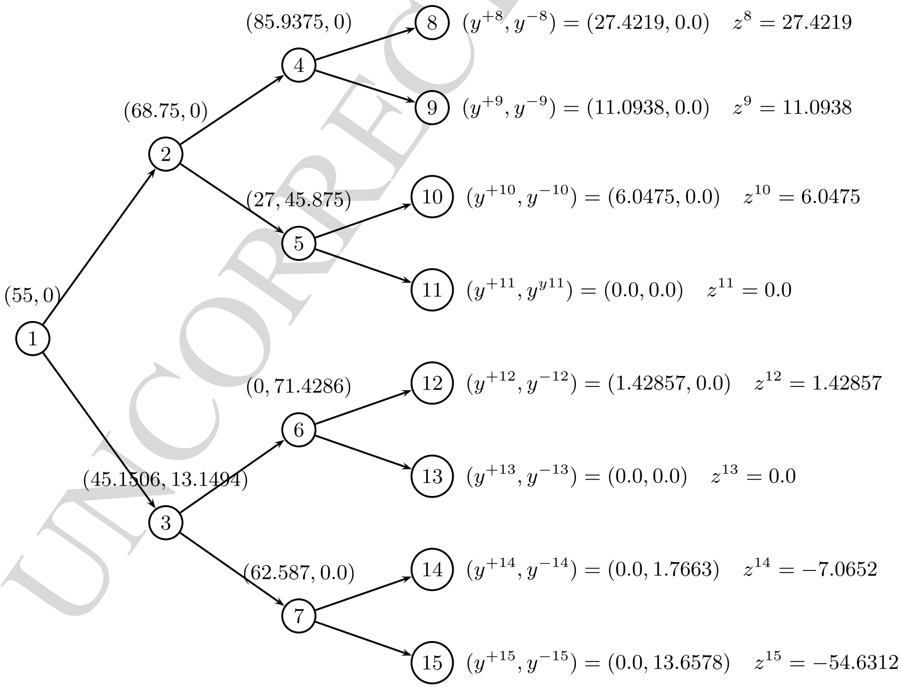

Fig. 3 Solution of the average scenario model at the first stage

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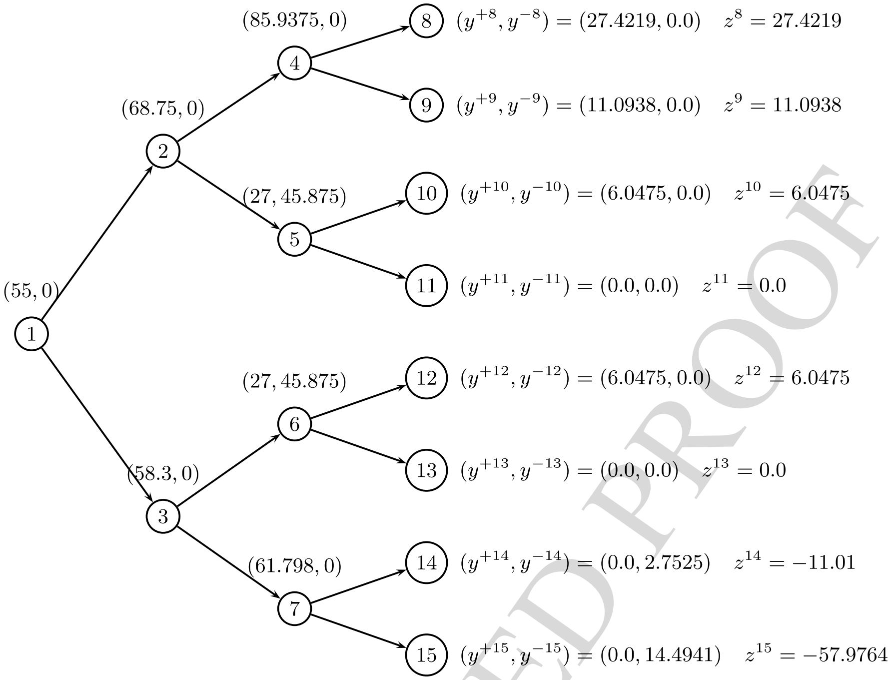

Fig. 4 Feasible solution of the average scenario model over the two first stages

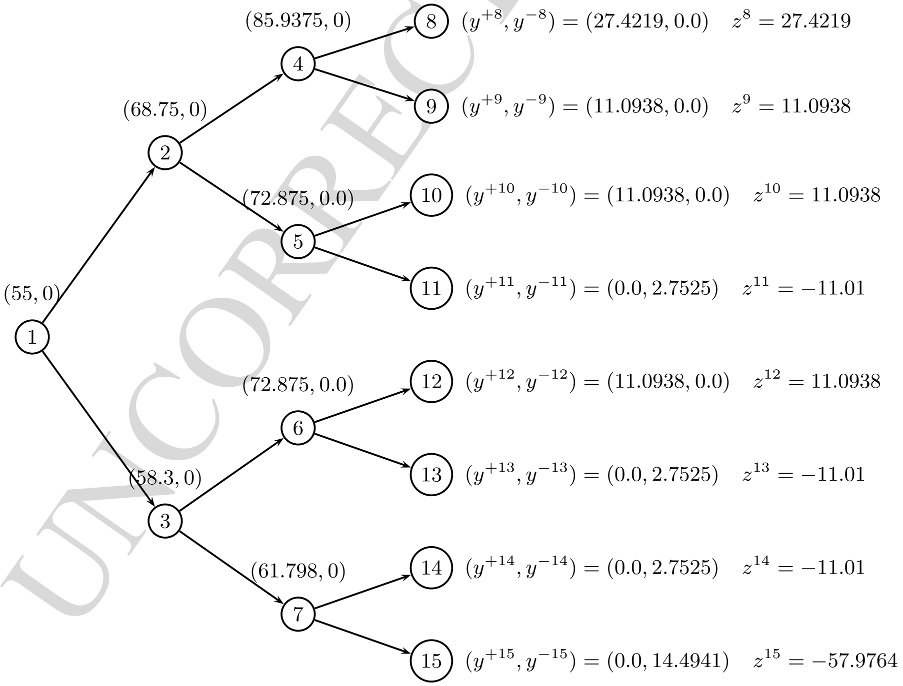

Fig. 5 Feasible solution of the average scenario model over the three first stages

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implementation of the feasible optimal solution of the average scenario over the three stages. Then $EDEV_4 = E\hat{E}V_4 = -3.78792$, which may be obvious since the scenario tree in this example is perfectly symmetric and balanced.

However, an additional conclusion is the absence of infeasibilities in the solution of the chain of problems studied. We introduce in the model defined for t exactly the values proposed by the optimal solution for $t-1$ and not merely the selection between different assets.

Finally, it results that

$$\begin{array}{ccccccccc}
WS & \geq & RP = EEV_1 & \geq & EEV_2 & \geq & EEV_3 & \geq & EEV_4 \\
10.5 & \geq & -1.5148 & \geq & -1.9631 & \geq & -\infty & \geq & -\infty \\
 & & & & & \geq & E\hat{E}V_3 & \geq & E\hat{E}V_4 \\
 & & & & & \geq & -2.29698 & \geq & -3.78792 \\
 & & & & & & & & \| \\
 & & EV = EDEV_1 & \geq & EDEV_2 & \geq & EDEV_3 & \geq & EDEV_4 \\
 & & 4.74 & \geq & 1.40235 & \geq & -1.7235 & \geq & -3.78792
\end{array}$$

defines a chain of intermediate expected values between the stochastic solution, RP, the expected result in each time period using the solution for the average scenario, EEV_t, and the value promised by the deterministic solution, EV.

Moreover, we compare it with the optimal solution under the expected perfect information, $WS = 10.5$. Notice that because of the structure of the problem, with uncertainty in the recourse matrix, $WS \leq EV$ is not satisfied. Also, the inequalities from Proposition 2 are not valid. However, Corollary 1 and Proposition 6 are verified. Thus,

$$VSS_1 = 0 \leq VSS_2 = 0.4483 \leq VSS_3 = 0.78218 \leq VSS_4 = 2.27312$$
$$VSS_3^D = 0.2087 \leq VSS_4^D = 2.27312.$$

In particular, we observe $VSS_3^D \leq VSS_3$, that is the goodness of the stochastic solution until the third stage is smaller when we compare it to the dynamic solution of the average scenario, EV_g, than when we compare it to the optimal solution of the average scenario, EV. Notice that, in this case, the deterministic dynamic model of the average scenario results in a better approximation to the stochastic model than the "classical" deterministic average scenario model until the third stage, at least.

6 Conclusions

In this paper, we generalize the definition of bounds for the optimal values of the objective function for various deterministic equivalent models in multistage stochastic linear programs. In particular, we introduce a chain of expected values when fixing the value of the decision variables at the optimal value in the related average scenario model, EV. The final value of the chain happens to be the expected value of using the expected value solution, EEV, in two-stage models. Differences between the values in this chain indicate the need to solve the stochastic model, RP. In each stage, they allow us to compute the value of the stochastic solution, VSS, and to check how

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good the approximation of the stochastic program by the deterministic one is up to the stage, when the expected values are used instead of the random variables. In order to motivate the definition of these bounds, we have used the problem of the private investor, which is well known in the relevant literature. The extension of the bounds that is proposed is primarily useful for avoiding to obtain the *RP* value when the average based solution is good enough, as it also happens for the two-stage problem.

Appendix. Some computational insight

In this paper, we have introduced the notion of expected result in t from using the expected value solution, EEV_t, and the value of the stochastic solution in t, VSS_t.

As we have seen, the value of EEV_t can be calculated by solving the *RP* model with additional constraints in order to fix the value of the decision variables until stage t. In the subset of problems to be solved, say $EEV_1, \ldots, EEV_T$, the complexity or difficulty of the solution increases when we are close to $EEV_1 = RP$, with this being the hardest model to solve. The information that they provide is also progressive in the same sense.

Otherwise, it is obvious that the *RP* model with fixed variables in the first $t-1$ stages is separable by the scenario groups of stage t. Then the value EEV_t can be computed as the following sum:

$$EEV_t = \sum_{g \in \mathcal{G}_t} Z^g, \tag{8}$$

where Z^g represents the optimal solution value of the model defined for each scenario group of stage t, $g \in \mathcal{G}_t$, in which the decision variables for the stages $1, 2, \ldots, t-1$, $(x_1, x_2, \ldots, x_{t-1})$, respectively, are fixed at the optimal values obtained in the solution of the average scenario problem (4).

That is, the value of EEV_t can be obtained from expression (8) breaking it down into the sum of $|\mathcal{G}_t|$ independent submodels, one for each group $g \in \mathcal{G}_t$, $t = t(g)$, such that

$$\begin{aligned}
Z^g = \max \sum_{\omega \in \Omega_g} w^\omega \Bigg[\sum_{\tau=1}^{t-1} \big(a_\tau^\omega \overline{x}_\tau + b_\tau^\omega y_\tau^\omega\big) + \sum_{\tau=t}^{T} \big(a_\tau^\omega x_\tau^\omega + b_\tau^\omega y_\tau^\omega\big) \Bigg] \\
\text{s.t.}\quad & B_\tau'^\omega y_{\tau-1}^\omega + B_\tau^\omega y_\tau^\omega = b_\tau^\omega - A_\tau'^\omega \overline{x}_{\tau-1} - A_\tau^\omega \overline{x}_\tau && \forall \omega \in \Omega_g, \forall \tau = 1, \ldots, t-1, \\
& A_\tau^\omega x_\tau^\omega + B_\tau'^\omega y_{\tau-1}^\omega + B_\tau^\omega y_\tau^\omega = b_\tau^\omega - A_\tau'^\omega \overline{x}_{\tau-1}, && \forall \omega \in \Omega_g, \tau = t, \\
& A_\tau'^\omega x_{\tau-1} + A_\tau^\omega x_\tau^\omega + B_\tau'^\omega y_{\tau-1}^\omega + B_\tau^\omega y_\tau^\omega = b_\tau^\omega, && \forall \omega \in \Omega_g, \tau = t+1, \ldots, T, \\
& x_\tau^\omega = x_\tau^{\omega'} \quad \forall \omega, \omega' \in \Omega_g \cap \Omega_f, \omega \neq \omega', f \in \mathcal{G}_\tau, \tau = t, \ldots, S, \\
& y_\tau^\omega = y_\tau^{\omega'} \quad \forall \omega, \omega' \in \Omega_g \cap \Omega_f, \omega \neq \omega', f \in \mathcal{G}_\tau, \tau = 1, \ldots, S, \\
& x_\tau^\omega \in X \quad \forall \omega \in \Omega_g, g \in \mathcal{G}_t, \tau = t, \ldots, T, \\
& y_\tau^\omega \in Y \quad \forall \omega \in \Omega_g, g \in \mathcal{G}_t, \tau = 1, \ldots, T.
\end{aligned}$$

The alternative modeling considered in (5) can be broken down in the same way.

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