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Javier Fernández-Macho[1] and Petr Mariel
Departamento de Econometría y Estadística
and Instituto de Economía Pública
Euskal Herriko Unibertsitatea-Universidad del País Vasco

Departamentos de
Econometría y Estadística (E.A. III),
Fundamentos del Análisis Económico,
Hacienda y Sistemas Fiscales (E.A. II)
e Instituto de Economía Pública

eman ta zabal zazu

Universidad del País Vasco
Euskal Herriko Unibertsitatea

Facultad de Ciencias Económicas.
Avda. Lehendakari Aguirre, 83
48015 BILBAO.

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Abstract

This paper proposes a new Hausman-like (H) test for the null of cointegration based on the efficient estimation of a cointegration regression and the subsequent consistent estimation of a regression in differences without making specific assumptions about the short-run dynamics of the data generating process. It is shown that, asymptotically, the H statistics are distributed as a standard chi-squared and are not affected by the inclusion of deterministic components in the regression, thus offering a simple way of testing for cointegration under the null. Besides, small sample critical values for these statistics are tabulated using Monte Carlo simulation and it is shown that these "not residual-based" tests exhibit appropriate size even for quite general error dynamics and good power against non-cointegrated alternatives. In fact, simulation results suggest that they perform quite reasonably when compared to some other —residual-based— tests of the null of cointegration.

Key words: *JEL Classification:* C22, C12.

* *Correspondence to:* J. Fernández-Macho, Dpto. de Econometría y Estadística, Fac. de Ciencias Económicas (UPV/EHU), Lehendakari Agirre 83, E48015 BILBAO, Spain. `e-mail (internet): jf@bs.ehu.es`. Financial aid from the University of the Basque Country (UPV-EHU) under grant UPV038.321-HA052/94 is gratefully acknowledged.

† *Correspondence to:* J. Fernández-Macho, Dpto. de Econometría y Estadística, Fac. de Ciencias Económicas (UPV/EHU), Lehendakari Agirre 83, E48015 BILBAO, Spain. `e-mail (internet): jf@bs.ehu.es`. Financial aid from the University of the Basque Country (UPV-EHU) under grant UPV038.321-HA052/94 is gratefully acknowledged.

1 Introduction

In recent times —since (Granger 1981, Granger 1983) introduced the notion of cointegrating relationships— testing for cointegration has acquired a great deal of importance in the empirical analysis of economic time series. As a result, quite a number of tests have been proposed for this purpose.

Some of the cointegration tests most widely used in practice are in fact already available unit-root tests (Dickey-Fuller and augmented Dickey-Fuller tests, Phillips' Z_α and Z_t statistics, Choi (1994)'s DHS, *etc.*) applied to the residuals from the cointegrating regression —the 'two-step' procedure suggested by Engle & Granger (1987). As these statistics are designed to test the presence of a unit root against a stationary alternative, when used on regression residuals cointegration appears as the alternative hypothesis rather than the null.

The null of cointegration seems a more natural choice since tests (say residual-based unit root tests) typically tend to find in favour of the null hypothesis (non-cointegration) unless there is considerable evidence to the contrary, but until now there have been very few attempts to test the cointegration hypothesis directly. Phillips & Ouliaris (1990) argue that the source of the difficulties lies in the failure of conventional asymptotic theory under the null of cointegration. However some simple ways to overcome this problem have been found within the family of residual-based tests itself (see *e.g.* Leybourne & McCabe (1994) and Shin (1994) and our discussion in section 4). Other related tests include that of Hansen (1992) who derived the large sample distribution of LM tests for parameter stability against several alternatives in the context of cointegrated regression models. In particular, testing for intercept stability against the alternative of a random-walk intercept —while the rest of the coefficients are held constant— would effectively be a test of the null of cointegration. However, Hansen (1992)'s test was not designed for this purpose and its actual alternative does not imply an $I(1)$ error process. (See also Quintos & Phillips (1992)). Park (1990) proposed two statistics for cointegration testing (J1 and J2 statistics for the nulls of cointegration and non-cointegration respectively). Both tests are based on the addition of some 'superfluous' regressors. If the variables in the underlying model are cointegrated, a standard testing procedure should be able to detect the superfluous nature of the added regressors —as compared to the 'true' ones. On the other hand, we would not expect this when the variables under consideration are not cointegrated and the relationship is itself spurious. Shin (1994) says that this test is rather *ad hoc* and indeed it remains unclear how to select the superfluous regressors.

The present paper adds further to this in the sense that it investigates a class of cointegration tests (under the null) which are not directly based on the residuals but on the estimated regression coefficients themselves. The statistics can be thought as based on the same simple principle as Hausman's specification test: here a regression in first differences being used as a benchmark for the (cointegrating) regression in levels.

The simplicity of the calculations is the first advantage of these statistics. Their second advantage is that they are asymptotically distributed as a chi-squared. The third advantage comes from the fact that the statistics' behaviour is not affected, at least asymptotically, by the inclusion of deterministic terms in the cointegrating regression. Finally, the study —through Monte Carlo simulation— of finite sample properties of these new Hausman-like statistics suggests that their nominal sizes remain quite unaltered by changes in the error dynamics while enjoying good power for a wide range of alternatives.

The plan of the paper is as follows. Section 2 presents the new Hausman-like test of cointegration under the null. Their asymptotic properties are examined in section 3. Finally section 4 presents critical values in finite samples together with Monte Carlo evidence on size and power comparisons.

2 The test

Consider the $(n+k)$-dimensional time series $(z_t', x_t')'$ $(t = 1 \dots T)$ generated by the following data generating process (dgp)

$$\begin{aligned} \underset{(n)}{z_t} &= \underset{(n\times d)}{\alpha}\ \underset{(d)}{g(t)} + \underset{(n\times k)}{\beta}\ \underset{(k)}{x_t} + \delta\, \underset{(n)}{u_t} + (1-\delta)\, \underset{(n)}{v_t}, \\ x_t &= x_{t-1} + \eta_t, \qquad u_t = u_{t-1} + v_t, \end{aligned} \tag{1}$$

where the elements of vector $g(t)$ are deterministic functions of time (such as time trends), $\zeta_t = (v_t', \eta_t')'$ follows an $(n+k)$-dimensional stationary process with zero mean and autocovariance function (acvf) $E(\zeta_t \zeta_{t-s}') = C(s)$, $(s = 0, \pm 1, \pm 2, \dots)$. In this system the scalar δ will be set to 0 if $(z_t', x_t')'$ is to be cointegrated (the null) in the sense of Engle & Granger (1987) (with $\beta \neq 0$) and $\delta = 1$ under the alternative (with $\beta = 0$ if $(z_t', x_t')'$ represent unrelated generalized random walks or $\beta \neq 0$ if they are related through their increments).

Let us assume that the acvf of ζ_t is absolutely summable (*i.e.* $\sum_{s=-\infty}^{\infty} \|C(s)\| < \infty$, where $\|\cdot\|$

is the Euclidean norm) and suppose that its spectral density $f(\cdot)$ is nowhere singular in $[-\pi, \pi]$. Following, say, (Brillinger 1975, p.296), we may then start by conditioning v_t on $\{\eta_t\}$ so that

$$v_t = \sum_{s=-\infty}^{\infty} \gamma_s \eta_{t-s} + \xi_t \tag{2}$$

where the $(n \times k)$ filter $\{\gamma_s\}$ is absolutely summable, *i.e.* $\sum_{s=-\infty}^{\infty} \|\gamma_s\| < \infty$, and ξ_t is an n-dimensional zero-mean stationary process such that $E(\xi_t \eta'_{t-s}) = 0$, $(s = 0, \pm1, \pm2, \ldots)$, $\forall t$. Therefore, (*c.f.* Saikkonen (1991)) $\exists m$ large enough so that $\gamma_s \approx 0$ for $|s| > m$ and the sum in (2) may be truncated at $|s| = m$.

Under the null of cointegration —$\delta = 0$ in (1)— the distribution of z_t conditional on $\{x_{t-m}, \ldots, x_{t+m}\}$ can then be written as

$$z_t = \alpha g(t) + \beta x_t + \sum_{s=-m}^{m} \gamma_s \eta_{t-s} + \varepsilon_t. \tag{3}$$

where $\varepsilon_t = \sum_{|s|>m} \gamma_s \eta_{t-s} + \xi_t \approx \xi_t$ (see the appendix). OLS estimation of (3) will produce an efficient (and superconsistent) estimator of the cointegrating vectors whose limiting distribution is free of the nuisance parameters γ_j arising from the short run dynamics of the dgp (Saikkonen 1991).

Alternatively, we notice that the asymptotic covariance matrix of x_t and $\{\eta_{t-m}, \ldots, \eta_{t+m}\}$ is block-diagonal (Phillips & Hansen 1988) which suggests estimating the set of nuisance parameters $\{\gamma_{-s} \ldots \gamma_s\}$ from a regression of the OLS residuals $\tilde{v}_t$ from (1) on $\{\eta_{t-s} = \Delta x_{t-s},\ |s| < m\}$, (where Δ is the difference operator) and then re-estimate β from $y_t = \beta x_t + \varepsilon_t$, where the regressand

$$y_t \overset{def}{=} z_t - \tilde{\alpha} g(t) - \sum_{s=-m}^{m} \tilde{\gamma}_s \Delta x_{t-s}$$

by construction, being $\tilde{\alpha}$ the vector of OLS estimates of coefficients of deterministic components and $\{\tilde{\gamma}_s\}$ the estimates of the nuisance parameters in (3).

Model (1) may thus be rewritten as

$$\begin{aligned} y_t &= \beta x_t + (\delta u_{t-1} + \varepsilon_t), \\ \Delta x_t &= \eta_t, \qquad \Delta u_t = v_t, \end{aligned} \tag{4}$$

where we recall that $\{\varepsilon_t\}$ is a stationary zero-mean process (asymptotically) uncorrelated with the increments of $\{x_t\}$ at all leads and lags while δ takes a zero value whenever the observed multivariate time series $(z'_t, x'_t)'$ is cointegrated and takes a value of one otherwise.

Under cointegration ($\delta = 0$) the error term is simply $\varepsilon_t \sim I(0)$, while under the alternative of no cointegration ($\delta = 1$) the error term becomes $(u_{t-1} + \varepsilon_t) \sim I(1)$. As a consequence, the OLS estimator $\hat{\beta}_l$ from the levels regression (4) will be T-consistent under the null of cointegration (Stock 1987) but it will have a nondegenerate distribution under the alternative.

On the other hand, taking differences (*i.e.* imposing one unit root)

$$\Delta y_t = \beta \eta_t + \varepsilon_t^{\star}, \tag{5}$$

where the error process $\{\varepsilon_t^{\star} = \delta v_{t-1} + \Delta\varepsilon_t\}$ is stationary always so that standard asymptotics on stationary variables apply yielding a $\sqrt{T}$-consistent under the null and an $O_p(T^{-1/2})$ estimator under the alternative (asymptotically biased since $E(\eta_t v_{t-1}')$ may not be equal to zero in general.) This regression in differences may then be used as a benchmark for the regression in levels in order to test for cointegration. The fact that we are able to reformulate our dgp (1) —through use of a suitable time domain correction— into regression (4) where the regressors x_t are made strictly exogenous, has important consequences for the applicability of our testing procedure since, otherwise, the regression in differences would in general be inconsistent under the null and it would not serve as a benchmark. Alternatively, instrumental variables could be used —as in Phillips & Hansen (1990). However, the existence of such cannot be taken for granted and a more general setup is desirable.

The presence of $g(t)$ in equation (1) implies 'stochastic' cointegration around some deterministic function of time. On the other hand, absence of any deterministic component means that there exist 'deterministic' cointegration in the sense that deterministic components are also eliminated together with the stochastic components. However, it is well known that the inclusion of deterministic components in the cointegration regression causes shifts in the asymptotic distributions of residual-based tests. This will not be so in our case since the regression in differences (5) —apart of being free of short-run-dynamics nuisance parameters— is also free, by construction, of deterministic components and, in consequence, our testing procedure will not be affected by them.

The so called Hausman test statistic (Hausman 1978, Durbin 1954), rests on the comparison between two estimators, both of them consistent under the null but with different probability limits under the alternative. The standardized difference between the two estimates will then have zero probability limit under the null but will diverge under the alternative (for test consistency). Accordingly a testing procedure based on the difference $c = \text{vec}(\hat{\beta}_d - \hat{\beta}_l)$ between the OLS estimators

Table 1: **Computation of the Hausman-like test statistics.**

1. OLS regression: $z_t = \alpha g(t) + \beta x_t + \sum_{s=-m}^{m} \gamma_s \Delta x_{t-s} + \varepsilon_t$ to obtain estimates $\tilde{\alpha}$, $\hat{\beta}_l$, $\hat{V}_l$, estimates of nuisance parameters $\{\tilde{\gamma}_{-s} \cdots \tilde{\gamma}_0 \cdots \tilde{\gamma}_s\}$, and the residual covariance matrix $\hat{V}_\varepsilon = \text{Var}(\varepsilon)$.

2. calculate y_t defined as $y_t = z_t - \tilde{\alpha} g(t) - \sum_{s=-m}^{m} \tilde{\gamma}_s \Delta x_{t-s}$

   alternatively...

   (a) OLS regression: $z_t = \alpha g(t) + \beta x_t + v_t$ to obtain estimates $\tilde{\alpha}$ and residuals $\{\tilde{v}_t\}$

   (b) OLS regression: $\tilde{v}_t = \sum_{s=-m}^{m} \gamma_s \Delta x_{t-s} + \varepsilon_t$ to obtain estimates of nuisance parameters $\{\tilde{\gamma}_{-s} \cdots \tilde{\gamma}_0 \cdots \tilde{\gamma}_s\}$

   (c) calculate y_t defined as $y_t = z_t - \tilde{\alpha} g(t) - \sum_{s=-m}^{m} \tilde{\gamma}_s \Delta x_{t-s}$

   (d) OLS regression of y_t on x_t (in levels): $y_t = \beta x_t + \varepsilon_t$ to obtain the estimates $\hat{\beta}_l$, $\hat{V}_l$ and the residuals covariance matrix $\hat{V}_\varepsilon = \text{Var}(\varepsilon)$
   and then...

3. OLS regression of Δy_t on Δx_t (in differences): $\Delta y_t = \beta \eta_t + \varepsilon^\star_t$ to obtain the estimates $\hat{\beta}_d$ and $\hat{V}_d$ with $\text{Var}(\varepsilon^\star) = D \hat{V}_\varepsilon D'$ where D is the $(T-1) \times T$ matrix whose (i,j)-th element is $d_{ij} = \begin{cases} -1, & i = j, \\ 1, & i = j-1, \\ 0, & \text{otherwise.} \end{cases}$

4. calculate the difference $c = \text{vec}(\hat{\beta}_d - \hat{\beta}_l)$ and the H statistics from (6).

obtained from a regression in first differences and a regression in levels can be proposed. The test statistics are:

$$\text{H1} = c'(\hat{V}_d + \hat{V}_l)^{-1} c, \qquad \text{H2} = c' \hat{V}_d^{-1} c, \tag{6}$$

where $\hat{V}_d$ and $\hat{V}_l$ are consistent estimates of the covariance matrices of $\hat{\beta}_d$ and $\hat{\beta}_l$ respectively. Since $\hat{V}_l$ is $O_p(T^{-2})$ while $\hat{V}_d$ is $O_p(T^{-1})$ both statistics are asymptotically equivalent. Table 1 summarizes the steps involved in the process of calculating the statistics.

Note also that H2 may be reinterpreted as the typical chi-squared statistic for testing whether $\hat{\beta}_d$ is significantly different from true β. In order to evaluate the statistic, the unknown β is replaced by $\hat{\beta}_l$ whose faster consistency rate ensures that the asymptotic distribution remains unaltered. The addition of $\hat{V}_l$ in the denominator of H1 may help provide a better approximation in small samples.

3 The asymptotic distributions

With respect to the error process $\{\varepsilon_t\}$ in the levels regression (4), let $C_\varepsilon(s) = E(\varepsilon_t \varepsilon'_{t-s})$ denote its acvf following the same convention as before for process $\{\zeta_t\}$. It will be convenient to identify the respective variances (at $s = 0$) as

$$\Sigma_\varepsilon \equiv C_\varepsilon(0); \qquad \Sigma = \begin{bmatrix} \Sigma_v & \Sigma'_{\eta v} \\ \Sigma_{\eta v} & \Sigma_\eta \end{bmatrix} \equiv C(0).$$

Let us also define the covariance matrix V_ε as the $(nT \times nT)$ Toeplitz matrix formed by $(n \times n)$ blocks with $C_\varepsilon(|i-j|)$ in the (i,j)'th position. As usual, all limits apply as $T \to \infty$, $\sum$ runs from $t = 1$ to T unless otherwise stated and the integral $\int B$ refers to the Lebesgue measure in the $(0,1]$ interval: $\int_0^1 B(r)dr$. Finally, it should be mentioned that "$\otimes$" is the Kronecker product and vec(A) is the nm column vector obtained by stacking the columns of an $(n \times m)$ matrix A one underneath the other.

As in Park & Phillips (1988) we require that the partial sums of $\{\zeta_t\}$ satisfy a multivariate invariance principle

$$T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} \zeta_t \Rightarrow B(r), \qquad r \in (0,1]$$

where "$\Rightarrow$" means weak convergence of the associated probability measures and $B(r) = (B_v(r)', B_\eta(r)')'$ denotes an $(n+k)$-variate Brownian motion with covariance matrix

$$\Omega = \lim_{T\to\infty} \mathrm{Var}(T^{-1/2} \sum_{t=1}^{T} \zeta_t) = \begin{bmatrix} \Omega_v & \Omega'_{\eta v} \\ \Omega_{\eta v} & \Omega_\eta \end{bmatrix} = 2\pi f_\zeta(0)$$

where $B(r)$ and Ω have been partitioned conformably with ζ_t. Note that $\Omega > 0$ since we may recall that the spectral density is nonsingular within $(-\pi, \pi)$.

Similarly, the partial sums of $\{\varepsilon_t\}$ in (4) will be such that

$$T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} \varepsilon_t \Rightarrow B_\varepsilon(r), \qquad r \in (0,1]$$

where $B_\varepsilon(r) = B_v - \Omega'_{\eta v} \Omega_\eta^{-1} B_\eta$ is a k-variate Brownian motion (uncorrelated by construction with B_η —and therefore independent of) with covariance matrix (Saikkonen 1991, p.11)

$$\Omega_\varepsilon = \Omega_v - \Omega'_{\eta v} \Omega_\eta^{-1} \Omega_{\eta v} = 2\pi f_\varepsilon(0)$$

where $f_\varepsilon(\cdot)$ is the error spectral density in (3) which is related to the spectral density of ζ_t through the expression $f_\varepsilon(\cdot) = f_v(\cdot) - f_{\eta v}(\cdot)' f_\eta(\cdot)^{-1} f_{\eta v}(\cdot)$ (Brillinger 1975, p.296).

Since Ω is nonsingular, we have also that $\Omega_\varepsilon > 0$. Note that $\Omega_v > 0$ rules out multicointegration (Granger & Lee 1989) and that $\Omega_\eta > 0$ rules out cointegration among the regressors.

Proposition 1 *In the multivariate regression model (1) under the null of cointegration*

$$\sqrt{T}c \overset{a}{\sim} \mathcal{N}(0, V_d),$$

$$Tc'V_d^{-1}c \overset{a}{\sim} \chi^2(nk), \qquad Tc'(V_d + T^{-1}V_l)^{-1}c \overset{a}{\sim} \chi^2(nk),$$

where $V_d = (\Sigma_\eta^{-1} \otimes I_n)R(\Sigma_\eta^{-1} \otimes I_n)$, and $V_l = (\int M(B_\eta)'M(B_\eta))^{-1} \otimes \Omega_\varepsilon$, with $R = \text{plim}\, T^{-1}(N'D \otimes I_n)V_\varepsilon(D'N \otimes I_n)$, $N' = (\eta_1, \ldots, \eta_{T-1})$ and $M(B)$ stands for 'standard', 'demeaned' or 'detrended' Brownian motion depending on whether $g(t) = 0$, 1 or $(1, t)$; that is, depending on whether there are no deterministic components in the cointegrating regression, it is a regression with a constant or it is a regression with a linear trend (see the appendix).

This proposition justifies the two suggested Hausman-like test statistics. Obviously, the effect of correcting the asymptotic variance of c adding the $T^{-1}V_l$ term disappears asymptotically but it nevertheless may provide a better approximation in small samples.

Proposition 2 *In the multivariate regression model (1) under the null of cointegration, for both Hausman-like statistics*

$$\text{H1}, \text{H2} \overset{a}{\sim} \chi^2(nk)$$

while under the alternative of no cointegration

$$T^{-1}\text{H1} = O_p(1) \qquad T^{-1}\text{H2} = O_p(1)$$

Note that, although asymptotically equivalent under the null, under the alternative H1 and H2 would not have the same limit distribution. This different asymptotic behaviour under the alternative may have consequences for test power: indeed in the limit $T^{-1}\text{H1} < T^{-1}\text{H2}$ and test H2 will be asymptotically more powerful.

In sum, it has been shown that the proposed Hausman-like test statistics are $O_p(1)$ under the null hypothesis of cointegration but they are $O_p(T)$ under the alternative, which ensures test consistency. Furthermore it has been shown that under cointegration the Hausman-like statistic tends asymptotically towards the standard chi-square distribution. Asymptotic tests can thus been performed straight away. All this means that the test statistics proposed may constitute a useful procedure for testing directly the hypothesis of cointegration under the null.

4 Small sample evidence

Critical values:

Tables 2 and 3 give single equation critical values of the statistics H1 and H2 calculated *via* Monte Carlo simulation. The data generating process ($DGP(1)$) was regression model (1) with $\delta = 0$, $g(t) = 0$, $\beta_1 = \ldots = \beta_k = 1$ and $\zeta_t \sim \text{iid}\mathcal{N}(0, I_{k+1})$. All series y_t, $x_{1t} \ldots x_{kt}$ thus generated are clearly $I(1)$ and they are cointegrated with cointegrating vector $(1, -1, \ldots, -1)$.

Using $DGP(1)$ the fractiles of the small sample distribution of H1 and H2 for $k = 1$ to 4 regressors and different sample sizes from $T = 10$ to 500 were approximated out of $100,000$ replications using the random number generator available with the RATS statistical package version 4.10.

It may be worth noticing how the finite sample distributions approach their corresponding asymptotic χ^2 distribution as $T \to \infty$ (see also figures 1 and 2). This approach —from below— is very smooth for H1, but not quite so for H2 due to a sort of adjustment process as a result of the term $T^{-1}V_l$ —which is not insignificant in small samples— having been dropped from its denominator while the numerator is the same.

Size and power comparisons:

Quite a few residual tests for the null of no-cointegration are already available such as the augmented Dickey-Fuller, Phillips $\hat{Z}_\alpha$ and $\hat{Z}_t$ (Phillips & Ouliaris 1990) or Durbin-Hausman tests (Choi 1994). Such tests are generally used as a benchmark to compare the power of new tests but due to the different nature of the null and alternative hypothesis of these test respect to H1 and H2 simulation-based comparisons are not straightforward. On the other hand, as mentioned in the introduction, the offer of tests which used cointegration as the null is not so rich as the offer of tests which have cointegration as the alternative. Finally, we chose Leybourne & McCabe (1994) and Shin (1994) residual-based tests of the null of cointegration for the comparison.

Leybourne & McCabe (1994)'s residual-based LBI test of cointegration is obtained as an extension of their previous LBI test for coefficient constancy (see Leybourne & McCabe (1989)) while Shin (1994)'s C test is a residual-based test obtained as an extension of an LM test of univariate stationarity (Kwiatkowski, Phillips, Schmidt & Shin 1992). It is easy to see that both pairs of proposals are actually identical except for some minor detail. They all use the same stochastic components model which —apart of a deterministic trend component and additional regressors as in (3)— can be written as follows (following Leybourne & McCabe (1994))

$$y_t = \alpha_t + \beta' x_t + \varepsilon_t, \qquad \alpha_t = \alpha_{t-1} + v_t, \tag{7}$$

where ε_t is stationary and v_t is iid$(0, \sigma^2)$ and independent of ε_t while $\alpha_0 \equiv \alpha$. Then they test the null hypothesis that α_t is not a random walk ($\sigma^2 = 0$). That means $\{\mu_t \equiv \alpha_t + \varepsilon_t\}$ is $I(0)$ under the null while it is $I(1)$ under the alternative. (In a related issue Shephard (1993) and Fernández-Macho (1993) investigate the small sample properties of respectively time domain and spectral estimates of the signal-to-noise ratio of $\{\mu_t\}$.)

If $\{S_t\}$ denotes the partial sum process of the OLS residuals from the cointegrating regression and $s^2(\ell)$ is a consistent semiparametric estimator of the long-run variance of the regression error, then their test statistic for cointegration is

$$C = \frac{\sum_{t=1}^{T} S_t^2}{T^2 s^2(\ell)}.$$

The limiting distribution of this statistic is a functional of Brownian motion and its critical values are calculated via Monte Carlo simulation and tabulated in Shin (1994). We may note that (Leybourne & McCabe 1994)'s LBI cointegration test in fact has the same form: the only difference being that in the estimation of the residual long-run variance

$$s^2(\ell) = T^{-1} \sum_{t=1}^{T} e_t^2 + 2T^{-1} \sum_{s=1}^{\ell} w(s, \ell) \sum_{t=s+1}^{T} e_t e_{t-s},$$

they use a rectangular lag window ($w(s, \ell) = 1$ if $|s| \leq \ell$ and 0 otherwise) while the former uses a Barlett's triangular lag window ($w(s, \ell) = 1 - s(\ell + 1)^{-1}$ if $|s| \leq \ell$ and 0 otherwise) as in Newey & West (1987), which guarantees the nonnegativity of $s^2(\ell)$. In this case using ℓ as a function of T was suggested by Kwiatkowski et al. (1992) following Schwert (1989): $\ell_j = \lfloor j(T/100)^{1/4} \rfloor$, with $j = 0, 4, 12$. Similarly for LBI, we used $j = 0, 2, 4$. We note that Andrews (1991)'s recommendation on the lag truncation parameter always gives a large value of ℓ for highly autocorrelated errors ($\phi > 0.8$ say), in which case the power of these tests is very poor. In practice the selection of the value of ℓ is rather critical, as already mentioned in Kwiatkowski et al. (1992) and Shin (1994). This will be quite apparent also in what follows.

Table 4 provides some evidence on the size and power of cointegration tests H1, H2, C and LBI in finite samples. It has been obtained from Monte Carlo simulation using 5,000 replications of sample size = 100 from a bivariate regression model (1) with $\beta = 1$ and

$$\text{DGP(2)} \left\{ \begin{array}{lcl} \eta_t & = & 0.45\eta_{t-1} + \varepsilon_{1t} + 0.35\varepsilon_{1,t-1} \\ v_t & = & (1-\delta)\phi v_{t-1} + \varepsilon_{2t} + \delta\theta\varepsilon_{2,t-1} \end{array} \right. \qquad \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \sim \text{iid}\mathcal{N}\left[0, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right]. \tag{8}$$

This setup allows for quite general behaviour. It turns out that the regressor $\{x_t\}$ follows an $ARIMA(1,1,1)$ process while the error term follows either a stationary $AR(1)$ if $\delta = 0$ or an $IMA(1,1)$ if $\delta = 1$. Besides, x_t is not exogenous because η_t and v_t exhibit nonzero correlations (at different lags). The parameter values chosen for the regressor generating process and the regressor-error correlation are high enough to offer a clear departure from both purely random errors and strictly exogenous regressors.

The two $I(1)$ series z_t and x_t generated by DGP(2) with $\delta = 0$ are cointegrated as long as $|\phi| < 1$. Therefore $(1-\phi)$ can be taken as a measure of how far we are from $\mathcal{H}_a$: non-cointegration. We chose $\phi \in \{0.2, 0.4, 0.6, 0.8, 1\}$ so that the evolution of the tests' size can be observed as the alternative is being approached. (Note that, as $\phi \to 1$, $\{v_t\}$ becomes a random walk.)

On the other hand, with $\delta = 1$, both z_t, x_t generated by DGP(2) are also $I(1)$ but they are not cointegrated as long as $\theta \neq -1$. Therefore $(1+\theta)$ can be taken as a measure of how far we are from $\mathcal{H}_0$: cointegration. We chose $\theta \in \{0, \pm 0.2, \pm 0.4, \pm 0.6, \pm 0.8, \pm 1\}$ so that the evolution of the tests' power can be observed as the null is being approached (note that, as $\theta \to -1$, $\{u_t\}$ collapses to white noise) as well as when it gets farther away (positive values). Note also that $\phi = \delta = 0$ and $\theta = -\delta = -1$ are identical cases within $\mathcal{H}_0$ while case $\theta = 0$ with $\delta = 1$ is equivalent to case $\phi = 1$ with $\delta = 0$ within $\mathcal{H}_a$. Figure 3 shows graphically all the cases involved in the comparisons.

As far as test size is concerned, the results reported indicate that in the presence of moderately autocorrelated errors ($0 < \phi \leq 0.6$ say) all tests —except $C(\ell_0)$ and $C(\ell_4)$: they suffer too soon from a serious overrejection problem which renders them rather useless in practice— maintain size distortions well within reasonable levels, although, as expected, for highly autocorrelated errors (in the range of $\phi = 0.8$ and higher) they will reject more often than their nominal size would indicate. This size distortion is perhaps not too severe (about twice as much at the 10% and 5% levels) for a sample of $T = 100$. C(ℓ_{12}) and LBI merit particular mention since they seem to reject too seldom at the 5% and 1% levels even when error autocorrelation is low. This is, probably, less due to a deficiency of the tests than to their use of tabulated critical values from the asymptotic distributions. However, for the very same reason, we might expect higher overrejection values than as reported in table 4 if the true small sample critical values were used.

The last two columns of table 4 correspond to the case $\phi = 1$ (which falls just outside the cointegration region) and the case $\theta = 1$ (the farthest away alternative considered). In fact $\{y_t\}$ and $\{x_t\}$ are two random walk processes which are correlated (through η_t); they represent the interesting practical case in which variables are related through their changes but not through their levels —in the sense that a meaningful relationship in levels does not exist. The results presented clearly favour H1 and H2: ruling out $C(\ell_0)$ and $C(\ell_4)$ because of their extreme overrejection levels under the null, we can observe that the Hausman-like statistics are more powerful. For example, for samples of size $T = 100$ at the 5% significance level, H2 will reject between 68% and 72% of the times the (wrong) null hypothesis of a levels relationship in favour of a (true) relationship in changes. In the same circumstance C(ℓ_{12}) managed just 29% rejections (20% for $\theta = 1$) and LBI(ℓ_4) about 40%.[1]

Figure 4 shows graphically the respective size and power of the tests involved in our comparison for all the cases considered [2]. As far as test power is concerned, we may notice how —unlike $C(\ell_{12})$ or LBI— the H statistics are rather sensitive to changes in the error dynamics under the alternative and will be so inasmuch as the variance of the (stationary) errors from the regression in differences used as benchmark does not account for all the 'long-run' variance of errors from the spurious levels regression. It is interesting to note that the power of H statistics is in direct relationship with the 'distance' from $\mathcal{H}_0$: the farther we are from it (as $\theta \to +1$) the larger the power and *viceversa* (as $\theta \to -1$). Although such behaviour is of course very reasonable, the drop in power in the latter case seems rather sharp and it may cause the H test to be slightly less powerful than say LBI for values of θ close to -1. On the whole, however, the Hausman-like test statistic exhibits a very satisfactory behaviour: it is much more powerful for a wide range of alternatives considered without rejecting more often than it should under the null.

Finally, table 5 presents evidence of the tests' power in the independent random walks case. It has been obtained from simulations using 20,000 replications of different sample sizes from $T = 10$ to 500 from the following

$$\text{DGP(3)} \left\{ \begin{array}{rcl} x_t & = & x_{t-1} + \eta_t \\ z_t & = & z_{t-1} + v_t \end{array} \right. ; \qquad \begin{pmatrix} \eta_t \\ v_t \end{pmatrix} = \text{iid}\mathcal{N}(0, I_2).$$

[1] In a similar experiment conducted with an exogenous random walk regressor and $\phi = 1$ —also for samples of size $T = 100$ at the 5% significance level— H2 rejected about 80% of the time the (wrong) null hypothesis of a levels relationship in favour of a (true) relationship in changes, while C(l_{12}) and LBI managed just 31% and 47% rejections respectively. (Results and details available from the authors on request).

[2] For C statistics the figure graphs the cases with $m = 5$ additional regressors; values of other parameters as in figure.

Power for test statistics H1 and H2 is reported in the first two blocks of table 5. As expected, H2 works slightly better than H1 in this respect.

The selection of the value of ℓ is again shown to be very critical. Thus, for greater ℓ a large number of observations are needed to reach a reasonable power of statistic C. Indeed, power of C(ℓ_{12}) is rather low for small to moderate sample sizes. (The 'high' power exhibited by C(ℓ_0) is rather deceptive since its real sizes are in fact several times the nominal ones as shown in table 4 for $T = 100$. The same is true, to a smaller degree, for C(ℓ_4).)

The H statistics are again superior as far as power against the random walks alternative is concerned, this is especially so for small to moderate sample sizes: for $T = 50$ for instance, H2 rejected, at the 5% nominal level, nearly 61% of the cases as against just about 19% for LBI(ℓ_4) and a mere 15% for C(ℓ_{12}).

In sum, the H test has a standard (χ^2) asymptotic distribution and shows higher power for an ample range of alternatives. Besides, the nominal size remains quite unaltered by changes in the error dynamics. Therefore, we are inclined to suggest that the H test may be a useful tool when testing the null hypothesis of cointegration.

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Appendix

Let $\tilde{\alpha}$, and $\tilde{\gamma}_j$ be the OLS estimators of deterministic components coefficients and nuisance parameters obtained from (3), let $\hat{\beta}_l$ be the OLS estimator obtained from the levels regression (4), and let $\hat{\beta}_d$ be the OLS estimator obtained from the differences regression (5).

Correspondingly, the estimators of the variances of $\hat{\beta}_l$ and $\hat{\beta}_d$ are respectively

$$\hat{V}_l = (X'X)^{-1} \otimes \hat{\Omega}_\varepsilon$$

$$\hat{V}_d \quad = \quad ((\Delta X)'(\Delta X)^{-1} \otimes I_n)\hat{R}((\Delta X)'(\Delta X)^{-1} \otimes I_n)$$

where $\hat{\Omega}_\varepsilon$ is a consistent estimator of the 'long run variance' matrix $\Omega_\varepsilon = 2\pi f_\varepsilon(0)$ and

$$R = ((\Delta X)'D \otimes I_n)\hat{V}_\varepsilon(D'(\Delta X) \otimes I_n)$$

where $\hat{V}_\varepsilon$ is a consistent estimator of the variance-covariance matrix of $\{\varepsilon_t\}$. and $X' = (x_1, \ldots, x_T)$.

First of all we want model (1) to be rewritten as (4) where the regressors are strictly exogenous. If $\gamma_s = 0$ for $|s| > m$ we have that $\varepsilon_t = \xi_t$, and the error term in (3) is uncorrelated with $\{\eta_t\}$ at all leads and lags so that the regressors in x_t are strictly exogenous. In general, of course, we cannot assume that $\gamma_s = 0$ for $|s| > m$ fixed; so that, following Saikkonen (1991), in order to work out the asymptotic distribution of our test statistics we will require $m \to \infty$ with T at a suitable rate such that $m^3/T \to 0$ and $T^{1/2}\sum_{|s|>m}\|\gamma_s\| \to 0$ specify upper and lower rate bounds for m (see also Said & Dickey (1984).) □

Following the convention established by Park & Phillips (1988), it will be convenient to define functionals of Brownian motion such as

$$h_0(B, M) = (\int \mathrm{d}BM')(\int MM')^{-1}, \qquad h_a(B, M, \pi) = (\int BM' + \pi)(\int MM')^{-1},$$

$$M(B) = \begin{cases} B(r), & \text{if } g(t) = 0, \\ B^*(r) = B(r) - \int B, & \text{if } g(t) = 1, \\ B^{**}(r) = B(r) + (6r - 4)\int B + (6 - 12r)\int sB, & \text{if } g(t) = (1, t), \end{cases}$$

and

$$P(B) = \begin{cases} 1 - [\int B'(\int BB')^{-1}] & \text{if } g(t) = 1, \\ \begin{bmatrix} 1 - \frac{3}{2}r - [(B' - \frac{3}{2}\int sB')(\int BB' - 3\int sB\int sB')^{-1}](B(r) - 3r\int sB) \\ r - \frac{1}{2} - [(\int sB' - \frac{1}{2}\int B')(\int BB' - \int B\int B')^{-1}](B(r) - \int B) \end{bmatrix} & \text{if } g(t) = (1, t). \end{cases}$$

Also we will make use of the following lemma, adapted from (Park & Phillips 1988, lemma 2.1), about weak convergence results of sample moments of $(g(t), x_t, \varepsilon_t)$.

Lemma 1 $T^{-3/2}\sum x_t \Rightarrow \int B_\eta$, $T^{-5/2}\sum tx_t \Rightarrow \int rB_\eta$, $T^{-2}\sum x_t x_t' \Rightarrow \int B_\eta B_\eta'$, $T^{-3/2}\sum t\varepsilon_t \Rightarrow \int r\mathrm{d}B_\varepsilon$, $T^{-1}\sum x_t\varepsilon_t' \Rightarrow \int B_\eta \mathrm{d}B_\varepsilon'$.

The following lemma expresses in a concise form the limiting distributions of the least squares estimators in (3)

Lemma 2 *Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}_s$ be the OLS estimators obtained from (3). Then under the null*

$$\text{(a)} \qquad \mathrm{diag}(T^{1/2}, T^{3/2})(\tilde{\alpha} - \alpha) \Rightarrow h_0(B_\varepsilon, P(B_\eta)),$$

$$\text{(b)} \qquad T(\tilde{\beta} - \beta) \Rightarrow h_0(B_\varepsilon, M(B_\eta)),$$

$$\text{(c)} \qquad (\frac{T}{m})^{1/2} \sum_{s=-m}^{m} (\tilde{\gamma}_s - \gamma_s) = O_p(1).$$

while under the alternative

$$\text{(a}'\text{)} \qquad \mathrm{diag}(T^{-1/2}, T^{1/2})(\tilde{\alpha} - \alpha) \Rightarrow h_a(B_v, P(B_\eta)),$$
$$= O_p(1)$$

$$\text{(b}'\text{)} \qquad (\tilde{\beta} - \beta) \Rightarrow h_a(B_v, M(B_\eta), 0),$$
$$= O_p(1),$$

$$\text{(c}'\text{)} \qquad m^{-1/2} \sum_{j=-m}^{m} (\tilde{\gamma}_s - \gamma_s) = O_p(1)$$

Results (a) and (b) were first obtained by Phillips & Durlauf (1986) and Park & Phillips (1988) (See also (Shin 1994, lemma 1)). The order of probability in (c) was obtained by Saikkonen (1991).

Proof: Write (3) in compact form for $g(t) = (1, t)$:

$$\begin{aligned} z_t &= \beta^* x_t^* + \varepsilon_t + \delta u_{t-1}, \\ \text{where} \qquad x_t^{*\prime} &= (g(t)', x_t', \eta_{t+m}', \ldots, \eta_{t-m}') \\ \text{and} \qquad \beta^* &= (\alpha, \beta, \gamma_{-m}, \ldots, \gamma_m). \end{aligned}$$

Under the null ($\delta = 0$), we may define the scale matrix

$$L = \mathrm{diag}(T^{-1/2}, T^{-3/2}, T^{-1} I_k, T^{-1/2} I_k, \ldots, T^{-1/2} I_k).$$

As $(\varepsilon_t - \xi_t) = o_p(T^{-1/2})$ (Saikkonen 1991, lemma A5) it can be shown that applying lemma 1

$$L^{-1}(\tilde{\beta}^* - \beta^*)' = (L \sum x_t^* x_t^{*\prime} L)^{-1} (L \sum x_t^* \varepsilon_t') \Rightarrow Q_x^{-1} Q_{x\varepsilon}$$

where

$$\hat{Q}_x = \begin{bmatrix} 1 & T^{-2} \sum t & T^{-3/2} \sum x_t' & 0 \\ T^{-2} \sum t & T^{-3} \sum t^2 & T^{-5/2} \sum t x_t' & 0 \\ T^{-3/2} \sum x_t & T^{-5/2} \sum t x_t & T^{-2} \sum x_t x_t' & 0 \\ 0 & 0 & 0 & T^{-1} \sum \eta_t^* \eta_t^{*\prime} \end{bmatrix}$$

$$\Rightarrow \quad Q_x = \begin{bmatrix} \multicolumn{2}{c}{I_2} & \begin{matrix} \int B_\eta \\ \int r B_\eta \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \int B_\eta' & \int r B_\eta' & \int B_\eta B_\eta' & 0 \\ 0 & 0 & 0 & V_\eta \end{bmatrix}$$

$$
\begin{aligned}
\hat{Q}_{x\varepsilon} &= \left[T^{-1/2}\sum \xi_t, T^{-3/2}\sum t\xi_t, T^{-1}\sum \xi_t x_t', T^{-1/2}\sum \xi_t \eta_t^{*\prime}\right]' \\
&\Rightarrow \quad Q_{x\varepsilon} = \left[B_\varepsilon(1), \int r \mathrm{d}B_\varepsilon, \int \mathrm{d}B_\varepsilon B_\eta', C_{\varepsilon\eta}\right]'
\end{aligned}
$$

where $\eta_t^{*\prime} = (\eta_{t+m}', \ldots, \eta_{t-m}')$, $V_\eta = \mathcal{E}(\eta_t^* \eta_t^{*\prime})$ and $C_{\varepsilon\eta} = \text{plim}\, T^{-1/2}\sum \xi_t \eta_t^{*\prime}$, and it is understood that the terms corresponding to $g(t)$ are deleted when not applicable. After inverting matrix Q_η and rearranging we obtain the results given.

Under the alternative ($\delta = 1$), we define the scale matrices

$$
\begin{aligned}
L_1 &= \text{diag}(T^{1/2}, T^{-1/2}, I_k, I_k, \ldots, I_k) \\
L_2 &= \text{diag}(T^{-3/2}, T^{-5/2}, T^{-2}I_k, T^{-1}I_k, \ldots, T^{-1}I_k),
\end{aligned}
$$

and applying lemma 1

$$
L_1^{-1}(\hat{\beta}^* - \beta^*)' = (L_2 \sum x_t^* x_t^{*\prime} L_1)^{-1}(L_2 \sum x_t^* u_t') \Rightarrow Q_x^{-1} Q_{xu}
$$

where Q_x is as before and

$$
\begin{aligned}
\hat{Q}_{xu} &= \left[T^{-3/2}\sum u_t, T^{-5/2}\sum t u_t, T^{-2}\sum u_t x_t', T^{-1}\sum u_t \eta_t^{*\prime}\right]' \\
&\Rightarrow \quad Q_{\eta\varepsilon} = \left[\int B_v, \int r B_v, \int B_v B_\eta', \int B_v \mathrm{d}B_\eta', \ldots, \int B_v \mathrm{d}B_\eta'\right]'
\end{aligned}
$$

again in the understanding that the terms corresponding to $g(t)$ are deleted when not applicable. Inverting matrix Q_η in each case and rearranging leads us to the results given. □

Lemma 3 *the OLS estimators of β in (3) and in (4) are asymptotically equivalent under the null, but differ under the alternative by an amount of $O_p(1)$.*

Proof: Trivially under the null since from $\tilde{\alpha} \xrightarrow{p} \alpha$, and $\tilde{\gamma}_s \xrightarrow{p} \gamma_s$, we have that $y_t \equiv z_t - \tilde{\alpha} g(t) - \sum_{s=-m}^{m} \tilde{\gamma}_s \eta_{t-s} \xrightarrow{p} z_t - \alpha g(t) - \sum_{s=-m}^{m} \gamma_s \eta x_{t-s} = \beta x_t + \varepsilon_t$, and the first part of the lemma follows.

Indeed, transforming the regression equation in such a way as (4) amounts to —apart of correcting second order bias— demeaning and detrending the variables, and we know that its effect carries through in exactly the same fashion to the asymptotics.

Under the alternative, from lemma 2, $T^{-1/2}(\tilde{\alpha}_0 - \alpha_0) = O_p(1)$, $T^{1/2}(\tilde{\alpha}_1 - \alpha_1) = O_p(1)$, and $\sum_{s=-m}^{m}(\tilde{\gamma}_s - \gamma_s) = O_p(m^{1/2})$, and we have that $y_t \equiv z_t - \tilde{\alpha} g(t) - \sum_{s=-m}^{m} \tilde{\gamma}_s \eta_{t-s} = \beta x_t + (u_t^* + \varepsilon_t)$

where

$$u_t^* = u_{t-1} - (\tilde{\alpha} - \alpha)g(t) - \sum_{s=-m}^{m} (\tilde{\gamma}_s - \gamma_s)\eta_{t-s}.$$

Then

$$\begin{aligned} T^{-2}\sum x_t u_t^{*\prime} &= T^{-2}\sum x_t u_{t-1}' - T^{-3/2}\sum x_t T^{-1/2}(\tilde{\alpha}_0 - \alpha_0)' - T^{-2}\sum t x_t T^{1/2}(\tilde{\alpha}_1 - \alpha_1)' \\ &\quad - \sum_{s=-m}^{m} (T^{-2}\sum x_t \eta_{t-s}')(\tilde{\gamma}_s - \gamma_s) \\ &\Rightarrow \int M(B_\eta)B_v' + \pi_{\eta v}', \end{aligned}$$

which defines $\pi_{\eta v}$ implicitly, and the lemma follows. Also

$$\begin{aligned} T^{-2}\sum u_t^* u_t^{*\prime} &= T^{-2}\sum (u_{t-1} - (\tilde{\alpha} - \alpha)g(t))(u_{t-1}' - g(t)'(\tilde{\alpha} - \alpha)') \\ &= T^{-2}\sum u_{t-1}u_{t-1}' - T^{-2}[\sum u_{t-1}g(t)'](\tilde{\alpha} - \alpha)' - (\tilde{\alpha} - \alpha)T^{-2}[\sum g(t)u_{t-1}'] \\ &\quad +(\tilde{\alpha} - \alpha)T^{-2}[\sum g(t)g(t)'](\tilde{\alpha} - \alpha)' \\ &\Rightarrow \int B_v B_v' + \pi_v. \end{aligned}$$

where $\pi_v = 3 - \int B_v - \int B_v' - \int rB_v - \int rB_v'$. □

The following lemma establishes the asymptotic distributions of the OLS levels regression estimator and its variance

Lemma 4 *Under the null of cointegration*

$$\begin{aligned} T(\hat{\beta}_l - \beta) &\overset{a}{\sim} \mathrm{N}(0, V_l), \\ T^2\hat{V}_l &\Rightarrow V_l \end{aligned}$$

where $V_l = (\int M(B_\eta)M(B_\eta)')^{-1} \otimes \Omega_\varepsilon$ *; while under the alternative*

$$\begin{aligned} (\hat{\beta}_l - \beta) &\Rightarrow h_a(B_v, M(B_\eta), \pi_{\eta v}) \\ &= O_p(1) \\ T\hat{V}_l &\Rightarrow (\int M(B_\eta)M(B_\eta)')^{-1} \otimes \Theta_u \\ &= O_p(1) \end{aligned}$$

where $\Theta_u = (\int B_v B_v' + \pi_v) - h_a(B_v, M(B_\eta), \pi_{\eta v})(\int M(B_\eta)B_v' + \pi_{\eta v}')$, so that $\hat{\beta}_l$ is a T-consistent estimator under the null but there is a stochastic asymptotic bias under the alternative.

Proof: writing model (4) in matrix form

$$Y = X\beta' + \delta U^* + E$$

where $Y' = (y_1 \dots y_T)$; $X' = (x_1 \dots x_T)$; $U^{*\prime} = (u_1^* \dots u_T^*)$; $E' = (\varepsilon_1 \dots \varepsilon_T)$; we obtain

$$\begin{aligned} T(\hat{\beta}_l - \beta) &= (\delta T^{-1} U^{*\prime} X + T^{-1} E' X)(T^{-2} X' X)^{-1}) \\ &\Rightarrow \delta T h_a(B_v, M(B_\eta), \pi_{\eta v}) + h_0(B_v, M(B_\eta)). \end{aligned}$$

Under the null ($\delta = 0$)

$$T(\hat{\beta}_l - \beta) \Rightarrow (\int \mathrm{d}B_\varepsilon M(B_\eta)')(\int M(B_\eta)M(B_\eta)')^{-1} = h_0(B_\varepsilon, M(B_\eta)) \sim \mathrm{N}(0, (\int M(B_\eta)M(B_\eta)')^{-1} \otimes \Omega_\varepsilon).$$

The normal distribution is reached because $M(B_\eta)$ is a vector process independent of B_ε (Park & Phillips 1988, lemma 5.1).

On the other hand,

$$T^2 \hat{V}_l = (T^{-2} X' X)^{-1} \otimes \hat{\Omega}_\varepsilon \Rightarrow (\int M(B_\eta)M(B_\eta)')^{-1} \otimes \Omega_\varepsilon$$

so that $\hat{V}_l$ converges to V_l at a very fast rate.

Under the alternative ($\delta = 1$) $T(\hat{\beta}_l - \beta)$ diverges but

$$\begin{aligned} (\hat{\beta}_l - \beta) &\Rightarrow h_a(B_v, M(B_\eta), \pi_{\eta v}) = \Theta_l \\ &= O_p(1) \end{aligned}$$

so that $\hat{\beta}_l$ is inconsistent, as well as its variance estimator

$$T\hat{V}_l = (T^{-2} X' X)^{-1} \otimes T^{-1} \hat{\Omega}_{\hat{u}}$$

$$\begin{aligned} T^{-1}\hat{\Omega}_{\hat{u}} &\rightarrow T^{-2} \hat{U}^{*\prime} \hat{U}^* \\ &= T^{-2} U' U - (\hat{\beta}_l - \beta) T^{-2} X' X (\hat{\beta}_l - \beta) \\ &\Rightarrow (\int B_v B_v' + \pi_v) - h_a(B_v, M(B_\eta), \pi_{\eta v})(\int M(B_\eta) B_v' + \pi_{\eta v}') = \Theta_u \end{aligned}$$

which leads to the last result in the lemma. □

Next lemma establishes the asymptotic distributions of the OLS differences regression estimator and its variance

Lemma 5 *Under the null of cointegration*

$$T^{1/2}(\hat{\beta}_d - \beta) \overset{a}{\sim} \mathrm{N}(0, V_d),$$
$$T\hat{V}_d \Rightarrow V_d$$

where $V_d = (\Sigma_\eta^{-1} \otimes I_n) R (\Sigma_\eta^{-1} \otimes I_n)$ *with* $R = \text{plim } T^{-1}(N'D \otimes I_n) V_\varepsilon (D'N \otimes I_n)$*; while under the alternative*

$$\begin{aligned}(\hat{\beta}_d - \beta) &\overset{p}{\to} C_{\eta v}(1)\Sigma_\eta^{-1} \\ &= O_p(1) \\ T\hat{V}_d &\overset{p}{\to} V_d \\ &= O_p(1)\end{aligned}$$

with $R = \text{plim } T^{-1}(N' \otimes I_n) V_v (N \otimes I_n)$,

so that $\hat{\beta}_d$ is a $\sqrt{T}$-consistent estimator under the null but, in general, there is a nonstochastic asymptotic bias under the alternative, in sharp contrast with the stochastic nature of the bias in the levels regression.

Proof: writing model (5) in matrix form

$$DY = N\beta' + \delta DU^* + DE$$

where $DU^{*\prime} = (\Delta u_1^* \ldots \Delta u_T^*)$ and $\Delta u_t^* = v_{t-1} - (\tilde{\alpha}_1 - \alpha_1) - \sum_{s=-m}^{m} (\tilde{\gamma}_s - \gamma_s) \Delta \eta_{t-s}$. The OLS estimator takes the the form

$$(\hat{\beta}_d - \beta) = (\delta U^{*\prime} D' N + E' D' N)(N'N)^{-1} \tag{9}$$

Under the null ($\delta = 0$) we are interested in the form of the limiting distribution of

$$\text{vec}(T^{-1/2} E' D' N) = T^{-1/2}(N'D \otimes I_n)\varepsilon \tag{10}$$

where $\varepsilon = \text{vec}E' = (\varepsilon_1', \ldots, \varepsilon_T')'$. We note that $\mathcal{E}(\varepsilon\varepsilon') = V_\varepsilon$ defined in section 3 as the $(nT \times nT)$ covariance matrix of process $\{\varepsilon_t\}$.

Let us define $\bar{\varepsilon} = V_\varepsilon^{-1/2}\varepsilon$, where $V_\varepsilon = V_\varepsilon^{1/2}(V_\varepsilon^{1/2})'$. Then

$$\mathcal{E}(\bar{\varepsilon}\bar{\varepsilon}') = V_\varepsilon^{1/2} \mathcal{E}(\varepsilon\varepsilon')(V_\varepsilon^{1/2})' = I_{nT}$$

so that $\bar{\varepsilon}_t \sim \text{iid}(0,1)$. Let us further define the $(kn \times Tn)$ matrix $\bar{N}' = (N'D \otimes I_n)V_\varepsilon^{1/2}$ with

$$\text{plim } T^{-1}\bar{N}'N = R = \text{plim } T^{-1}(N'D \otimes I_n)V_\varepsilon(D'N \otimes I_n),$$

and $\mathcal{E}(\bar{N}_t\bar{\varepsilon}_t) = 0, \forall t$. Therefore, applying Mann-Wald theorem

$$\text{plim } T^{-1}\bar{N}'\bar{\varepsilon} = 0, \qquad T^{-1/2}\bar{N}'\bar{\varepsilon} \overset{a}{\sim} \mathcal{N}(0,R).$$

Substituting in (10) we get

$$\text{vec}(T^{-1/2}E'D'N) = T^{-1/2}\bar{N}'\bar{\varepsilon} \overset{a}{\sim} \mathcal{N}(0,R).$$

Finally, let plim $T^{-1}N'N = \Sigma_\eta > 0$ in (9). We may the write

$$\sqrt{T}(\hat{\beta}_d - \beta) \overset{a}{\sim} \mathcal{N}(0, (\Sigma_\eta^{-1} \otimes I_n)R(\Sigma_\eta^{-1} \otimes I_n)).$$

On the other hand,

$$\begin{aligned} T\hat{V}_d &= [(T^{-1}N'N)^{-1} \otimes I_n]T^{-1}\hat{R}[(T^{-1}N'N)^{-1} \otimes I_n] \\ T^{-1}\hat{R} &= (T^{-1}N'D \otimes I_n)\hat{V}_\varepsilon(T^{-1}D'N \otimes I_n) \end{aligned}$$

where $\hat{V}_\varepsilon \xrightarrow{p} V_\varepsilon$. Hence $T^{-1}\hat{R} \xrightarrow{p} R$ and $T\hat{V}_d \xrightarrow{p} V_d$.

Under the alternative $(\delta = 1)$ and (9) becomes

$$(\hat{\beta}_d - \beta) = [(DU^* + DE)'N](N'N)^{-1}$$

where

$$\begin{aligned} T^{-1}\sum \eta_t(\Delta u_t^{*\prime} + \Delta\varepsilon_t') &= T^{-1}\sum \eta_t v_{t-1}' - T^{-3/2}\sum \eta_t T^{1/2}(\tilde{\alpha}_1 - \alpha_1)' \\ &\quad - \sum_{s=-m}^{m}(T^{-1}\sum \eta_t \Delta\eta_{t-s})(\tilde{\gamma}_s - \gamma_s)' + T^{-1}\sum \eta_t \Delta\varepsilon_t' \\ &\xrightarrow{p} \mathcal{E}(\eta_t v_{t-1}') = C_{\eta v}(1). \end{aligned}$$

(Note that the bias coming from the inconsistency of $\tilde{\alpha}_1$ or $\tilde{\gamma}_s$ disappears asymptotically). Therefore

$$(\hat{\beta}_d - \beta) \xrightarrow{p} C_{\eta v}(1)\Sigma_\eta^{-1} = \Theta_d.$$

On the other hand

$$\begin{aligned} T\hat{V}_d &= [(T^{-1}N'N)^{-1} \otimes I_n]T^{-1}\hat{R}[(T^{-1}N'N)^{-1} \otimes I_n] \\ T^{-1}\hat{R} &= (T^{-1/2}N' \otimes I_n)\hat{V}_v(T^{-1/2}N' \otimes I_n) \end{aligned}$$

where $\hat{V}_v \xrightarrow{p} V_v$, the covariance matrix of process v_t. Hence $T^{-1}\hat{R} \xrightarrow{p} R$ and $T\hat{V}_d \xrightarrow{p} V_d$. □

Proof of proposition 1:

$$\begin{aligned}
\sqrt{T}c &= \text{vec}\sqrt{T}(\hat{\beta}_d - \beta) - T^{-1/2}\text{vec}T(\hat{\beta}_l - \beta) \\
&= (1, -T^{-1/2}) \begin{bmatrix} \sqrt{T}\text{vec}(\hat{\beta}_d - \beta) \\ T(\hat{\beta}_l - \beta) \end{bmatrix}.
\end{aligned}$$

From lemmas 4 and 5 $\sqrt{T}(\hat{\beta}_d - \beta) \overset{a}{\sim} \mathcal{N}(0, V_d)$, $T(\hat{\beta}_l - \beta) \overset{a}{\sim} \mathcal{N}(0, V_l)$. Therefore, their joint distribution is also gaussian. Let C_{dl} denote the asymptotic covariance matrix of $\sqrt{T}(\hat{\beta}_d - \beta)$ and $T(\hat{\beta}_l - \beta)$. Then we may write

$$\begin{bmatrix} \sqrt{T}(\hat{\beta}_d - \beta) \\ T(\hat{\beta}_l - \beta) \end{bmatrix} \overset{a}{\sim} \mathcal{N}\left[0, \begin{pmatrix} V_d & C_{dl}' \\ C_{dl} & V_l \end{pmatrix}\right]$$

and hence the limiting distribution of $\sqrt{T}c$ has zero mean and variance

$$\begin{pmatrix} 1 & -T^{-1/2} \end{pmatrix} \begin{pmatrix} V_d & C_{dl}' \\ C_{dl} & V_l \end{pmatrix} \begin{pmatrix} 1 \\ -T^{-1/2} \end{pmatrix} = V_d - T^{-1/2}(C_{dl} + C_{dl}') + T^{-1}V_l$$

where the last two terms disappear asymptotically and the proposition follows. □

Proof of proposition 2: Under the null, from lemmas 4 and 5 and proposition 1 the result follows trivially.

Under the alternative, from lemmas 4 and 5 and defining $\Theta_c = \Theta_d - \Theta_l$

$$c = \text{vec}(\hat{\beta}_d - \beta) - \text{vec}(\hat{\beta}_l - \beta) \Rightarrow \text{vec}\Theta_c.$$

Since Θ_l is a stochastic matrix but Θ_d is a matrix of constants, we have that $\Pr(\Theta_l = \Theta_d) = 0$ and plim $c \neq 0$ almost surely. In fact, it also follows that under the alternative c will have the same asymptotic distribution as $-\hat{\beta}_l$ except for a shift in the mean of value vecΘ_d. Then from lemma 5, $T\hat{V}_d \xrightarrow{p} (\Sigma_\eta^{-1} \otimes I_n)R(\Sigma_\eta^{-1} \otimes I_n)$; therefore

$$\begin{aligned}
T^{-1}\text{H2} &= T^{-1}c'\hat{V}_d^{-1}c = c'(T\hat{V}_d)^{-1}c \Rightarrow (\text{vec}\Theta_c\Sigma_\eta^{-1})'R(\text{vec}\Theta_c\Sigma_\eta^{-1}) \\
&= O_p(1),
\end{aligned}$$

and from lemma 4, $T\hat{V}_l \xrightarrow{p} (\int M(B_\eta)M(B_\eta)')^{-1} \otimes \Theta_u$; therefore

$$\begin{aligned}
T^{-1}\text{H1} &= T^{-1}c'(\hat{V}_d + \hat{V}_d)^{-1}c = c'(T\hat{V}_d + T\hat{V}_l)^{-1}c \\
&\Rightarrow (\text{vec}\Theta_c)'\left[(\Sigma_\eta^{-1} \otimes I_n)R(\Sigma_\eta^{-1} \otimes I_n) + (\int M(B_\eta)M(B_\eta)')^{-1} \otimes \Theta_u\right]^{-1} \text{vec}\Theta_c \\
&= O_p(1).
\end{aligned}$$

As $\int MM' > 0$ and $\Theta_u > 0$ we also have that plim $T^{-1}H1 <$ plim $T^{-1}H2$. □

Figure 1: **Empirical Distributions of Hausman-like Test Statistics (m=1).**

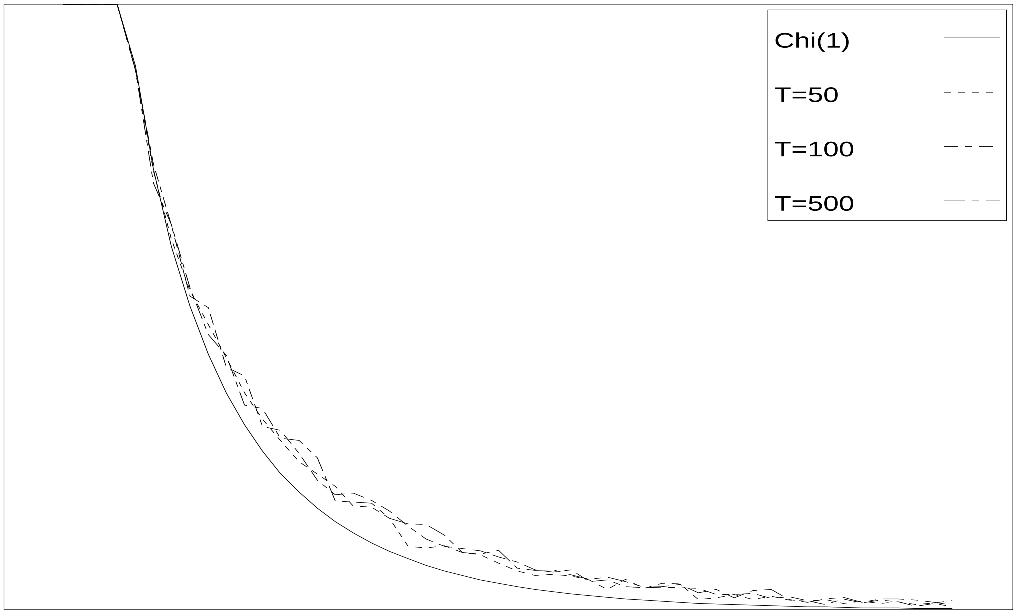

Figure 2: **Empirical Distributions of Hausman-like Test Statistics (m=4).**

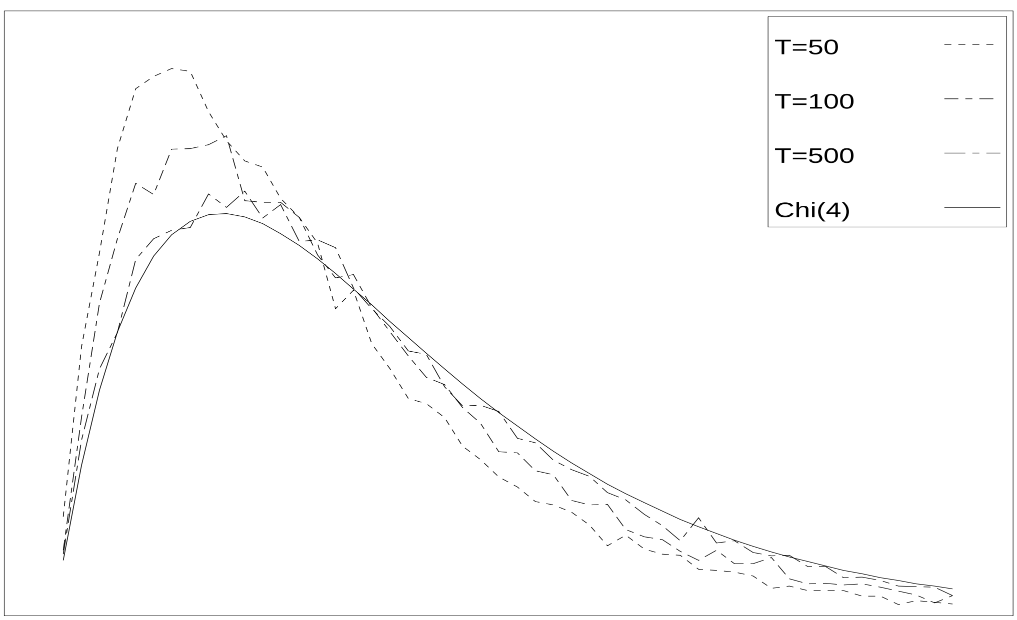

Figure 3: **DGP's for Size and Power.**

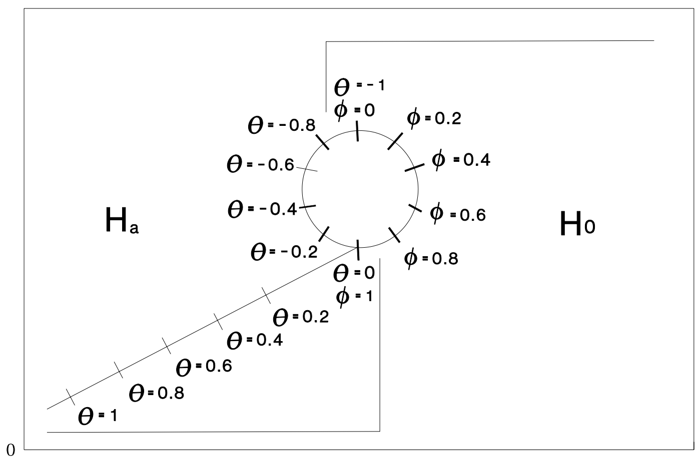

Figure 4: **Empirical Size and Power of Cointegration Tests. (T=100)**

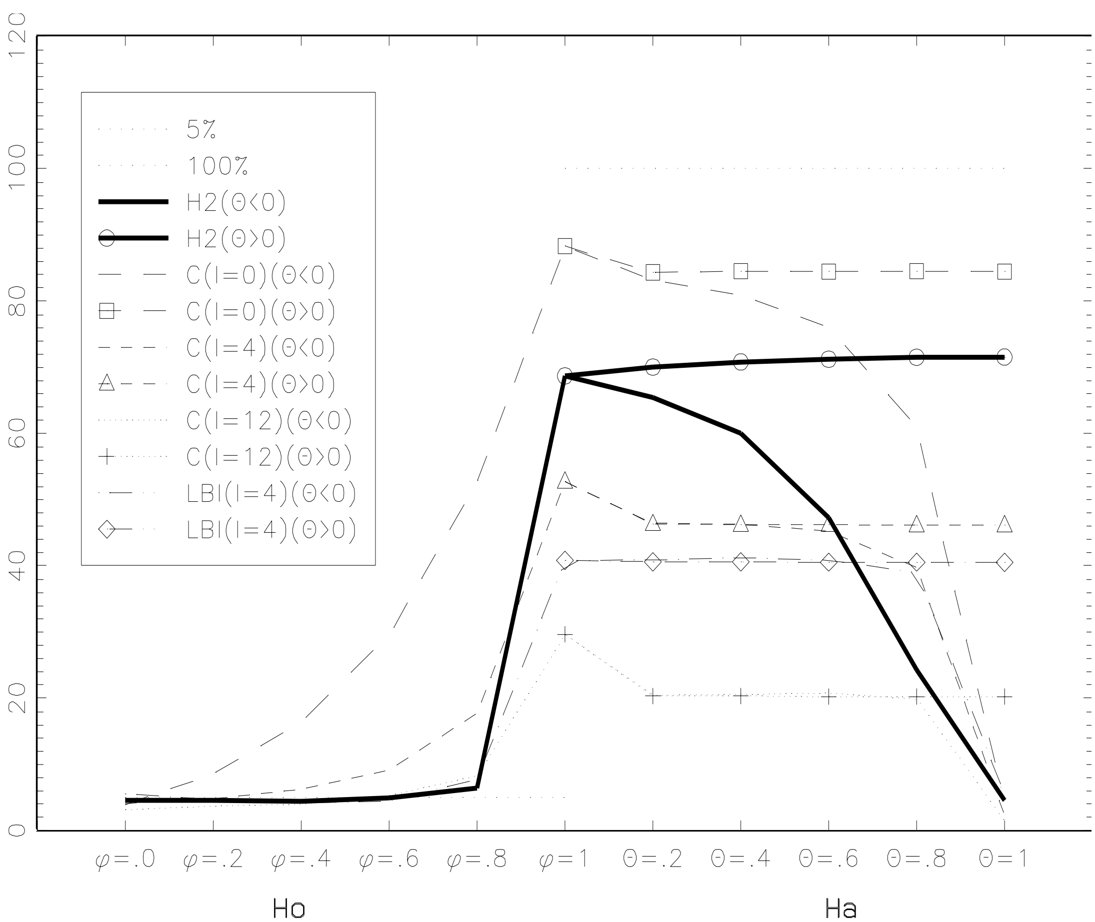

Table 2: **Critical Values for the $H1$ Statistic**

T	**k**	*Critical value* 0.25	0.50	0.75	0.90	0.95	0.99
10	*1*	0.067	0.306	0.936	2.063	3.081	5.921
	2	0.282	0.709	1.547	2.842	3.964	7.203
	3	0.449	0.944	1.829	3.151	4.314	7.674
	4	0.555	1.072	1.957	3.291	4.518	8.018
20	*1*	0.081	0.367	1.099	2.327	3.399	6.304
	2	0.383	0.949	1.975	3.462	4.686	7.852
	3	0.697	1.404	2.570	4.181	5.468	8.768
	4	0.965	1.762	3.001	4.658	5.965	9.337
30	*1*	0.087	0.388	1.152	2.431	3.520	6.373
	2	0.433	1.067	2.180	3.758	5.030	8.160
	3	0.823	1.645	2.946	4.699	6.055	9.332
	4	1.181	2.118	3.549	5.373	6.791	10.072
40	*1*	0.091	0.406	1.182	2.469	3.585	6.387
	2	0.464	1.130	2.313	3.939	5.238	8.352
	3	0.900	1.783	3.167	4.979	6.335	9.776
	4	1.319	2.347	3.896	5.799	7.250	10.735
50	*1*	0.091	0.413	1.209	2.508	3.605	6.465
	2	0.480	1.172	2.397	4.032	5.297	8.380
	3	0.946	1.874	3.320	5.156	6.558	9.908
	4	1.411	2.510	4.125	6.127	7.635	11.159

T	**k**	*Critical value* 0.25	0.50	0.75	0.90	0.95	0.99
100	*1*	0.097	0.435	1.268	2.613	3.713	6.464
	2	0.521	1.271	2.553	4.302	5.623	8.794
	3	1.062	2.094	3.666	5.629	7.118	10.475
	4	1.646	2.878	4.662	6.831	8.422	11.955
150	*1*	0.099	0.442	1.284	2.639	3.751	6.520
	2	0.538	1.305	2.630	4.403	5.779	9.076
	3	1.104	2.183	3.812	5.856	7.363	10.816
	4	1.734	3.034	4.896	7.129	8.752	12.512
200	*1*	0.099	0.447	1.297	2.637	3.732	6.469
	2	0.549	1.326	2.672	4.470	5.855	9.084
	3	1.128	2.216	3.891	5.937	7.467	10.999
	4	1.772	3.101	5.002	7.287	8.924	12.619
250	*1*	0.099	0.446	1.298	2.670	3.739	6.466
	2	0.552	1.343	2.677	4.498	5.920	9.175
	3	1.149	2.254	3.915	6.033	7.569	11.078
	4	1.800	3.142	5.078	7.369	9.021	12.702
500	*1*	0.101	0.449	1.301	2.670	3.762	6.537
	2	0.567	1.356	2.728	4.566	5.938	9.169
	3	1.176	2.305	4.021	6.126	7.671	11.135
	4	1.857	3.244	5.214	7.561	9.198	13.007
∞	*1*	0.102	0.455	1.323	2.706	3.841	6.635
	2	0.575	1.386	2.773	4.605	5.991	9.210
	3	1.213	2.366	4.108	6.251	7.815	11.345
	4	1.923	3.357	5.385	7.779	9.488	13.277

Table 3: **Critical Values for the $H2$ Statistic**

		Critical value								*Critical value*					
T	**k**	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*	*0.99*	**T**	**k**	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*	*0.99*
10	*1*	0.083	0.380	1.170	2.597	3.953	7.898	*100*	*1*	0.100	0.447	1.303	2.682	3.806	6.620
	2	0.370	0.930	2.032	3.791	5.366	10.353		*2*	0.545	1.328	2.663	4.492	5.882	9.188
	3	0.610	1.283	2.505	4.374	6.115	11.567		*3*	1.129	2.224	3.890	5.969	7.534	11.069
	4	0.766	1.483	2.752	4.785	6.749	13.144		*4*	1.777	3.102	5.023	7.349	9.058	12.868
20	*1*	0.092	0.413	1.236	2.620	3.823	7.102	*150*	*1*	0.101	0.450	1.307	2.682	3.816	6.626
	2	0.454	1.127	2.335	4.092	5.544	9.303		*2*	0.555	1.346	2.711	4.534	5.956	9.364
	3	0.859	1.728	3.157	5.132	6.709	10.772		*3*	1.151	2.276	3.979	6.108	7.667	11.263
	4	1.225	2.230	3.796	5.888	7.557	11.775		*4*	1.829	3.200	5.165	7.500	9.205	13.166
30	*1*	0.094	0.423	1.251	2.646	3.821	6.877	*200*	*1*	0.101	0.453	1.316	2.673	3.779	6.550
	2	0.492	1.205	2.467	4.240	5.667	9.229		*2*	0.561	1.358	2.733	4.570	5.984	9.285
	3	0.971	1.932	3.447	5.470	7.061	10.836		*3*	1.164	2.288	4.013	6.125	7.708	11.351
	4	1.427	2.557	4.261	6.429	8.129	12.122		*4*	1.846	3.231	5.212	7.579	9.288	13.122
40	*1*	0.097	0.432	1.262	2.633	3.820	6.811	*250*	*1*	0.100	0.450	1.313	2.701	3.779	6.532
	2	0.513	1.245	2.555	4.345	5.759	9.172		*2*	0.562	1.368	2.725	4.584	6.032	9.374
	3	1.023	2.030	3.597	5.642	7.175	10.979		*3*	1.180	2.314	4.016	6.187	7.758	11.369
	4	1.538	2.732	4.534	6.745	8.407	12.396		*4*	1.863	3.249	5.249	7.607	9.323	13.135
50	*1*	0.096	0.434	1.272	2.636	3.794	6.793	*500*	*1*	0.101	0.452	1.309	2.685	3.783	6.574
	2	0.521	1.272	2.595	4.376	5.736	9.079		*2*	0.572	1.370	2.754	4.607	6.001	9.236
	3	1.054	2.094	3.695	5.727	7.275	10.912		*3*	1.193	2.336	4.075	6.202	7.778	11.277
	4	1.610	2.861	4.690	6.969	8.655	12.605		*4*	1.889	3.300	5.308	7.684	9.356	13.208
								∞	*1*	0.102	0.455	1.323	2.706	3.841	6.635
									2	0.575	1.386	2.773	4.605	5.991	9.210
									3	1.213	2.366	4.108	6.251	7.815	11.345
									4	1.923	3.357	5.385	7.779	9.488	13.277

Table 4: **Empirical Size and Power of Cointegration Tests (T=100)**

			H_0: *Cointegration*				H_a: *No Cointegration*	
Stat.	m	**Sig. level**	$\phi = 0.2$	$\phi = 0.4$	$\phi = 0.6$	$\phi = 0.8$	$\phi = 1.0$	θ=1.0
H1		10%	8.76	9.00	9.24	11.52	70.20	72.56
		5%	4.52	4.26	4.78	6.10	65.20	67.16
		1%	0.92	0.92	1.18	1.70	55.18	57.94
H2		10%	8.74	9.00	9.24	11.78	73.36	75.98
		5%	4.48	4.32	4.84	6.38	68.64	71.48
		1%	0.90	0.96	1.16	1.92	59.62	63.06
C(ℓ_0)	0	10%	16.04	27.98	45.78	70.44	95.72	96.00
		5%	9.02	18.22	33.64	58.38	91.54	92.06
		1%	2.64	6.84	17.62	38.94	80.58	81.34
C(ℓ_4)	0	10%	11.06	13.92	18.70	30.32	66.96	66.56
		5%	5.26	6.62	10.26	20.22	58.04	57.14
		1%	0.62	1.28	2.50	6.32	40.14	39.46
C(ℓ_{12})	0	10%	10.92	11.86	13.36	17.92	46.02	45.40
		5%	3.88	4.50	5.66	8.62	33.20	32.02
		1%	0.10	0.20	0.26	0.80	9.02	7.98
C(ℓ_0)	5	10%	15.00	25.12	40.24	64.08	94.04	91.14
		5%	8.48	16.36	29.52	52.66	88.28	84.44
		1%	2.64	6.46	14.68	34.22	75.88	70.44
C(ℓ_4)	5	10%	9.70	11.92	16.06	27.36	62.74	56.40
		5%	4.76	6.20	9.06	17.66	52.72	46.14
		1%	0.94	1.24	2.38	6.50	35.52	27.66
C(ℓ_{12})	5	10%	9.44	10.04	12.04	16.82	42.80	34.24
		5%	3.64	3.94	5.24	8.22	29.60	20.18
		1%	0.08	0.16	0.30	0.48	7.26	0.48
LBI(ℓ_2)		10%	9.42	10.16	13.40	25.74	66.32	66.22
		5%	4.22	4.90	6.82	15.64	57.32	57.24
		1%	0.50	0.5	1.02	4.50	38.88	38.70
LBI(ℓ_4)		10%	10.56	9.56	9.98	16.04	52.60	52.30
		5%	4.62	4.08	4.38	7.72	40.80	40.46
		1%	0.48	0.36	0.28	1.06	18.94	18.76

Table 5: **Power of the statistics against independent random walks**

		Sample size									
Stat.	Sig. level	*10*	*20*	*30*	*40*	*50*	*100*	*150*	*200*	*250*	*500*
H1	10%	26.9	43.9	52.5	57.8	61.8	72.4	76.7	80.0	82.4	87.3
	5%	17.9	35.1	44.5	50.3	54.8	67.2	72.3	76.3	79.1	85.0
	1%	6.3	20.4	30.5	37.1	42.7	57.4	64.5	69.3	72.7	80.2
H2	10%	31.0	49.4	57.9	63.1	66.5	76.1	80.0	82.9	84.9	89.1
	5%	23.2	41.3	50.6	56.4	60.7	71.8	76.3	79.7	82.1	87.2
	1%	10.6	27.8	38.3	44.8	49.6	63.2	69.2	73.6	76.7	83.2
C(ℓ_0)	10%	44.8	64.3	75.6	82.3	86.4	96.6	98.6	99.5	99.7	99.9
	5%	32.9	54.4	65.6	73.3	78.8	92.4	96.7	98.4	99.1	99.9
	1%	12.3	35.5	48.4	57.2	63.2	81.7	89.9	93.6	96.0	99.4
C(ℓ_4)	10%	23.3	40.9	50.4	50.8	56.1	67.1	77.4	83.4	84.1	95.0
	5%	3.7	26.4	38.4	38.5	44.7	57.1	67.7	74.4	75.5	90.1
	1%	0.0	2.8	15.8	15.7	23.6	39.2	50.8	58.3	59.5	77.9
C(ℓ_{12})	10%	1.7	13.1	24.4	29.1	32.2	45.4	52.8	57.5	61.2	75.3
	5%	0.0	0.0	4.8	10.9	15.2	31.5	41.0	46.6	50.8	65.0
	1%	0.0	0.0	0.0	0.0	0.0	7.6	18.7	26.0	31.2	48.5
LBI(ℓ_2)	10%	20.2	38.3	48.5	56.3	61.5	66.2	76.8	83.0	87.4	96.5
	5%	3.0	24.1	30.7	44.7	51.3	56.2	67.0	73.9	79.7	92.6
	1%	0.2	2.1	14.4	24.0	31.8	38.1	50.3	57.9	63.8	81.9
LBI(ℓ_4)	10%	7.6	24.9	35.8	34.1	40.4	52.1	62.1	68.7	69.3	85.7
	5%	1.9	7.8	20.9	19.2	26.7	40.5	51.5	58.7	59.3	77.6
	1%	0.6	0.1	0.5	0.1	3.9	18.5	32.3	41.2	41.8	61.3