Hausman-like and Variance-ratio Tests for Cointegrated Regressions

Javier Fernández-Macho and Petr Mariel
Departamento de Econometría y Estadística
and Instituto de Economia Pública
Euskal Herriko Unibertsitatea-Universidad del País Vasco
Lehendakari Aguirre 83
E48015 BILBAO Spain
e-mail (internet): jf@alcib.bs.ehu.es

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Abstract

In this paper critical values for the Hausman-like and Variance-ratio tests statistics of the null of cointegration are presented. Besides the critical values, which are tabulated using simulation, the asymptotic distribution of these tests are derived. It is also shown that these tests substain good power against the independent random walks alternative and that they may do better in small samples than some residual-based test of the null of cointegration.

Key words: Brownian motion, common dynamic factors, unit root, Monte Carlo simulation, Wiener process.

1 Introduction

Several tests of cointegration have been proposed since Granger (1983) introduced the idea of cointegration. The majority of these tests test the null of no-cointegration against the alternative of cointegration. It is often argued that cointegration would be a more natural choice, but there are only a few tests for the null of cointegration (Shin (1993)).

The first objective of this paper is to present the critical values of three tests of the null of cointegration —and not residuals-based— which were first defined by Fernndez (1993). The first two tests are based on Hausman's specification test and the other ones are just ratios of variances of the cointegrating regression and regression in first differences.

The second objective of this paper is to compare the power of these tests with different tests of the null of cointegration such as Shin (1993) or Leybourne and McCabe (1993).

The model:

Let $(y_t, \mathbf{x}_t')'$ be a $(k+1 \times 1)$ vector of observed variables (from $t = 1$ to T) related via the following model

$$\begin{aligned} y_t &= \boldsymbol{\beta}'\mathbf{x}_t + u_t, \\ \mathbf{x}_t &= \mathbf{x}_{t-1} + \boldsymbol{\eta}_t, \end{aligned}$$

where $\boldsymbol{\eta}_t \sim \text{iid}(\mathbf{0}, \Sigma_\eta)$ and uncorrelated with $\{u_t\}$.

Under cointegration $\{u_t\}$ will be a stationary zero mean process with variance σ^2; while under the alternative of no cointegration $u_t \sim I(1)$. As a consequence, the OLS estimator $\hat{\boldsymbol{\beta}}_L$ will be T-consistent under the null of cointegration but inconsistent under the alternative.

On the other hand, taking differences (*i.e.* imposing one unit root)

$$\Delta y_t = \boldsymbol{\beta}'\boldsymbol{\eta}_t + u_t^\star,$$

where $u_t^\star \sim I(0)$ always, so that standard asymptotics on stationary variables apply now under both the null and the alternative yielding a $\sqrt{T}$-consistent estimator $\hat{\boldsymbol{\beta}}_D$ in both cases. Following Fernndez (1993) this regression on differences may then be used as a benchmark for the regression in levels in order to test for cointegration.

The statistics:

The so called Hausman test statistic (Hausman (1978), Durbin (1954)) rests on the comparison between two estimators, both of them consistent under the null but one of them inconsistent under the alternative. The difference between the two estimates will then have zero probability limit under the null but will diverge under the alternative (for test consistency).

Accordingly, Fernndez (1993) proposes a testing procedure based on the difference $\mathbf{c} = \hat{\boldsymbol{\beta}}_D - \hat{\boldsymbol{\beta}}_L$ between the OLS estimators obtained from two regressions in levels and in first differences respectively. The test statistics are:

$$\mathrm{H1} = \mathbf{c}'(\hat{\mathbf{V}}_D + \hat{\mathbf{V}}_L)^{-1}\mathbf{c}, \qquad \mathrm{H2} = \mathbf{c}'\hat{\mathbf{V}}_D^{-1}\mathbf{c}.$$

Since $\hat{\mathbf{V}}_L \sim O_p(T^2)$ while $\hat{\mathbf{V}}_D \sim O_p(T)$ both statistics are asymptotically equivalent, but there may be differnces in small samples.

Alternatively, rather than on direct comparison between both estimators, Fernndez (1993) proposes a variance-ratio statistic based on comparing some measure of spread for the estimators such as their respective generalized variances:

$$\mathrm{J} = T\sqrt[k]{|\hat{\mathbf{V}}_L|/|\hat{\mathbf{V}}_D|}.$$

2 The Asymptotic distributions

The asymptotic distributions of the relevant statistics involved can be illustrated for a very simple model with $k = 1$ regressors, *i.e.* the bivariate random-walk model

$$\begin{aligned} &y_t = \beta x_t + u_t; \quad H_0 : u_t \sim \mathrm{iid}(0, \sigma^2), \qquad H_a : \Delta u_t = a_t \sim \mathrm{iid}(0, \sigma_a^2), \\ &x_t = x_{t-1} + \eta_t; \quad \eta_t \sim \mathrm{iid}(0, \sigma_\eta^2) \quad \text{and uncorrelated with } \{u_t\} \text{ or } \{a_t\}\ . \end{aligned} \tag{1}$$

The model in (first) differences becomes

$$\begin{aligned} &\Delta y_t = \beta \Delta x_t + u_t^\star; \quad H_0 : u_t^\star \sim (0, 2\sigma^2), \quad H_a : u_t^\star = a_t, \\ &\Delta x_t = \eta_t \end{aligned}$$

where $\{u_t^\star\}$ is independent of $\{\eta_t\}$ but, under the null, it is serially correlated since it follows a (noninvertible) MA process.

We will discuss now the asymptotic behaviour of the relevant statistics under the null of cointegration. In what follows all the limits are taken as $T \to \infty$. The sum $\sum$ is taken to be from $t = 1$ to T. The functions $W(t)$, $U(t)$, and $A(t)$ denote standard Wiener processes (Brownian motions) resulting from mapping into the interval $[1, 0]$ the integrated processes $\sum \eta_t$, $\sum u_t$, and $\sum a_t$ respectively as $T \to \infty$. The symbol $\Rightarrow$ denotes weak convergence. (See appendix for demonstrations.)

Proposition 1 *In the bivariate random-walk model (1) under the null of cointegration* $\sqrt{T}c \overset{a}{\sim} N(0, V_D + T^{-1}V_L)$, [1] *where* $V_L = \frac{\sigma^2}{\sigma_\eta^2}(\int_0^1 W(t)^2 dt)^{-1}$ *and* $V_D = 2\sigma^2/\sigma_\eta^2$.

Therefore $\frac{Tc^2}{V_D + T^{-1}V_L} \overset{a}{\sim} \chi^2(1)$ which justifies the proposed Hausman-like test statistic.

Proposition 2 *In the bivariate random-walk model (1) under the null of cointegration, for both Hausman-like statistics*

$$H \Rightarrow \frac{1}{2}(\int dW\, d^2U)^2 \sim \chi^2(1)$$

while under the alternative of no cointegration

$$T^{-1}H1 \;\Rightarrow\; \left[\left(\frac{\int W^2 dt}{\int WAdt}\right)^2 + \frac{\int A^2 dt \int W^2 dt}{(\int WAdt)^2} - 1\right]^{-1} = O(1)$$

$$T^{-1}H2 \;\Rightarrow\; \left(\frac{\int WAdt}{\int W^2 dt}\right)^2 = O(1)$$

Therefore the limiting distribution of the test statistics H1 and H2 is a standard χ^2 distribution with one degree of freedom, whose critical values are widely available in ordinary statistical tables.

Note that, although asymptotically equivalent under the null, under the alternative H1 and H2 would not have the same limit distribution. This different asymptotic behaviour under the alternative may have consequences for test power: indeed as $\int A^2 \mathrm{d}t \int W^2 \mathrm{d}t > (\int WA\mathrm{d}t)^2$ then in the limit $T^{-1}\mathrm{H1} < T^{-1}\mathrm{H2}$ under the alternative and test H2 will be asymptotically more powerful.

Proposition 3 *In the bivariate random-walk model (1) under the null of cointegration, the variance-ratio statistic*

$$J \Rightarrow (2\int_0^1 W(t)^2 dt)^{-1}$$

while under the alternative of no cointegration

$$T^{-1}J \Rightarrow \left[\frac{\int A^2 dt \int W^2 dt - (\int WAdt)^2}{(\int W^2 dt)^2}\right] = O(1).$$

Therefore the limiting distribution of the test statistics J is such that $(2J)^{-1}$ converges to a functional whose distribution is given in Nabeya and Tanaka (1988, eq. 4.10) (see also Abadir 1992 [2]).

[1] Of course the second term in the variance would dissapear asymptotically but it has been left so as to provide a better aproximation in small samples.

[2] The same limiting distribution —except for a factor of 2— appears for one of Choi (1992)'s statistics, although, of course, the finite sample distributions differ substantially.

In sum, it has been shown that the proposed Hausman-like and variance-ratio test statistics are $O_p(1)$ under the null hypothesis of cointegration but they are $O_p(T)$ under the alternative, which ensures test consistency. Furthermore it has been shown that under cointegration the Hausman-like statistic tends asymptotically towards the standard chi-square distribution of one degree of freedom while the variance-ratio statistic tends towards a non-standard distribution given by

$$\mathcal{F}(J) = 2\sqrt{2}\sum_{j=0}^{\infty}\begin{pmatrix} -1/2 \\ j \end{pmatrix}\Phi(-(2j+1/2)\sqrt{2J})$$

where $\Phi(\cdot)$ is the standard normal distribution.

Asymptotic tests can thus been performed straight away. All this means that the test statistics proposed may constitute a useful procedure for testing directly the hypothesis of cointegration under the null.

3 Small sample evidence

Critical values:

The critical values of the statistics $H1$, $H2$, and J were calculated using simulation. The data generating process ($DGP(1)$) used for the Monte Carlo study is the following:

$$\begin{aligned} x_t &= x_{t-1} + \epsilon_{1t} \\ y_t &= x_t + \epsilon_{2t} \end{aligned}$$

$$\begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix} = \text{iid}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

Both series y_t, x_t are clearly $I(1)$ and are cointegrated with cointegrating vector $(1, -1)$. Using DGP(1) the fractiles of the limiting distribution of $H1$, $H2$, and J were simulated out of 100.000 iterations. The estimated fractiles are reported in tables 1, 2 and 3.

Power comparisons:

There are many tests for the null of no-cointegration such as augmented Dickey-Fuller, Phillips $\hat{Z}_\alpha$ and $\hat{Z}_t$ (Phillips and Outliaris 1990) or Durbin-Hausman tests (Choi 1994). These tests which are generally used to compare the power of new tests could not be used in our case, because the statistics $H1$, $H2$, and J have the null of cointegration. The offer of tests which used cointegration as the null is not so rich as the offer of tests which have cointegration as the alternative. We choose finally the Shin's residual based test of the null of cointegration (Shin 1993) for our comparision.

The Shin's test of cointegration is a residual based test which is an extension of the LM test of univariate stationarity (Kwiatowski *et al.* 1992). They use the following components model:

$$y_t = \alpha + \delta t + X_t, \qquad X_t = \gamma_t + v_t, \qquad \gamma_t = \gamma_{t-1} + u_t \tag{2}$$

where v_t is stationary and u_t is iid. Then they test the null hypothesis that X_t has no random walk error component ($\sigma_u^2 = 0$). That means X_t is $I(0)$ under the null.

Shin (1993) considers the following cointegrating regression with some additional terms:

$$y_t = Z_t'\beta + \sum_{j=-K}^{K} \Delta Z_{t-j}\pi_j + X_t \tag{3}$$

$$y_t = \alpha_\mu + Z_t'\beta_\mu + \sum_{j=-K}^{K} \Delta Z_{t-j}\pi_{\mu j} + X_{\mu t} \tag{4}$$

$$y_t = \alpha_\tau + \delta_\tau t + Z_t'\beta_\tau + \sum_{j=-K}^{K} \Delta Z_{t-j}\pi_{\tau j} + X_{\tau t} \tag{5}$$

where X_t hs the same form as defined in (2), while v_t and u_t are iid normal. The additive regressors ΔZ_t serve to relax the assumption of strict exogeneity of Z_t with respect to v_t which seems to be too restrictive in time series modelling. If S_μ, $S_{\mu t}$ and $S_{\tau t}$ are the partial sum processes of the OLS residuals ($\hat{X}_t$, $\hat{X}_{\mu t}$ and $\hat{X}_{\tau t}$) from the cointegrating regressions with the additive terms (3), (4) and (5) and $s^2(l)$ is a consistent semiparametric estimator of the long run variance of the regression error, then the test statistics for cointegration in (3),(4) and (5) are:

$$C = \frac{\sum_{t=1}^{T} S_t^2}{T^2 s^2(l)}, \qquad C_\mu = \frac{\sum_{t=1}^{T} S_{\mu t}^2}{T^2 s^2(l)}, \qquad C_\tau = \frac{\sum_{t=1}^{T} S_{\tau t}^2}{T^2 s^2(l)} \tag{6}$$

respectively. The limiting distributions of these statistics are functionals of Brownian motions and the critical values for C, C_μ and C_τ are calculated via Monte Carlo simulation and are tabulated in Shin (1993).

The DGP(2) for the power comparison is the following univariate model:

$$\begin{array}{rcl} x_t &=& x_{t-1} + \nu_{1t} \\ y_t &=& y_{t-1} + \nu_{2t} \end{array} \; ; \qquad \begin{pmatrix} \nu_{1t} \\ \nu_{2t} \end{pmatrix} = \text{iid}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

The power of the test statistics $H1$, $H2$ and J are reported in table 4. One thing to note is that the modified Hausman-like statistic $H2$ works slightly better than $H1$. The power of statistic J is the best of the three statistics considered —at least for DGP(2).

The power of the statistics C and C_μ for DGP(2) are reported in tables 5 and 6. (C_τ will not work better as there is no trend in DGP(2).) The three distinct panels in these tables correspond to three different windows in the estimation of $s^2(l)$. We used an estimator of the form:

$$s^2(l) = T^{-1} \sum_{t=1}^{T} e_t^2 + 2T^{-1} \sum_{s=1}^{l} w(s,l) \sum_{t=s+1}^{T} e_t e_{t-s},$$

the term $w(s,l)$ corresponding to Barlett spectral window $w(s,l) = 1 - s(l+1)^{-1}$ as in Newey and West (1987), which guarantees the nonnegativity of $s^2(l)$. We considered three possible values of l as a function of T following Schwert (1989): $l_0 = 0, l_4 = \lfloor 4(T/100)^{1/4} \rfloor$, and $l_{12} = \lfloor 12(T/100)^{1/4} \rfloor$.

The power of C and C_μ for l_0 is obviously difficult to beat and only J obtains similar performance. However the selection of the value of l is very critical in practice, as already mentioned in Kwiatowski *et al* (1992), and is demostrated with the 2nd and 3rd panel of tables 5 and 6. Thus, for grater l a large number of observations are needed to reach a reasonable power of the statistics C and C_μ. This is not the case for statistics $H1$, $H2$ and J which, besides, work rather well in smaller samples.

4 Conclusions

In this paper three alternative statistics for testing the null of cointegration are presented. These statistics use the results of estimation of the cointegrating regeression in levels and in first differences. In spite of their simplicity, the power of these statistics appears satisfactory, as is shown by power comparisons with other statistics suggested.

The majority of existing tests of cointegration test the null hypothesis of no cointegration. Combining these kind of tests with the tests proposed in this paper which have the null of cointegration can lead to more accurate conclusions.

Appendix

- $\hat{\beta}_L$ under the null:

$$\begin{aligned} T(\hat{\beta}_L - \beta) &= \frac{T^{-1} \sum x_t u_t}{T^{-2} \sum x_t^2} = \frac{T^{-1} \sum (T^{-1/2} x_t)(T^{1/2} u_t)}{T^{-1} \sum (T^{-1/2} x_t)^2} \\ &\Rightarrow \frac{\sigma}{\sigma_\eta} \left(\frac{\int_0^1 W(t) \mathrm{d}U(t)}{\int_0^1 W(t)^2 \mathrm{d}t} \right) \sim \mathrm{N}(0, V_L) \end{aligned}$$

where $V_L = \frac{\sigma^2}{\sigma_\eta^2} (\int_0^1 W(t)^2 \mathrm{d}t)^{-1}$ —Note that $\int W \mathrm{d}U \sim \mathrm{N}(0, \int W^2 \mathrm{d}t)$ implies $\frac{\int W \mathrm{d}U}{\int W^2 \mathrm{d}t} \sim \mathrm{N}(0, [\int W^2 \mathrm{d}t]^{-1})$.

- $\hat{\beta}_D$ under the null: Obviously $\hat{\beta}_D$ is not efficient under the null because $u_t^\star = \Delta u_t \sim$ noninvertible MA(1). Nevertheless

$$\begin{aligned}\sqrt{T}(\hat{\beta}_D - \beta) &= \frac{T^{-1/2}\sum(\Delta x_t)u_t^\star}{T^{-1}\sum(\Delta x_t)^2} = \frac{T^{-1/2}\sum \eta_t u_t^\star}{T^{-1}\sum \eta_t^2} \\ &\Rightarrow \frac{\sigma}{\sigma_\eta}(\underbrace{\int_0^1 \mathrm{d}W(t)\mathrm{d}^2U(t)}_{\mathrm{N}(0,2)}) \sim \mathrm{N}(0, V_D)\end{aligned}$$

  where $V_D = 2\sigma^2/\sigma_\eta^2$.

- $\hat{V}_L$ under the null:

$$\begin{aligned}T^2\hat{V}_L &= \frac{\hat{\sigma}^2}{T^{-2}\sum x_t^2} = \frac{T^{-1}\sum \hat{u}_t^2}{T^{-1}\sum(T^{-1/2}x_t)^2} \\ &\Rightarrow \frac{\sigma^2}{\sigma_\eta^2}(\int_0^1 W(t)^2\mathrm{d}t)^{-1} = V_L\end{aligned}$$

- $\hat{V}_D$ under the null:

$$\begin{aligned}T\hat{V}_D &= \frac{\hat{\sigma}_\star^2}{T^{-1}\sum(\Delta x_t)^2} = \frac{T^{-1}\sum \hat{u}_t^{\star 2}}{T^{-1}\sum \eta_t^2} \\ &\Rightarrow \frac{2\sigma^2}{\sigma_\eta^2} = V_D\end{aligned}$$

- $c = (\hat{\beta}_D - \hat{\beta}_L)$ under the null: (Of course the second term in the expressions below would dissapear asymptotically but it has been left so as to provide a better aproximation in small samples.)

$$\begin{aligned}\sqrt{T}c &= \sqrt{T}(\hat{\beta}_D - \beta) - \sqrt{T}(\hat{\beta}_L - \beta) \\ &\Rightarrow \frac{\sigma}{\sigma_\eta}(\int \mathrm{d}W\mathrm{d}^2U) - \frac{\sigma}{\sqrt{T}\sigma_\eta}\frac{\int W\mathrm{d}U}{\int W^2\mathrm{d}t} \\ &\Rightarrow \frac{\sigma}{\sigma_\eta}(\underbrace{\int \mathrm{d}W\mathrm{d}^2U}_{\mathrm{N}(0,2)} - \frac{1}{\sqrt{T\int W^2\mathrm{d}t}}\underbrace{\frac{\int W\mathrm{d}U}{\sqrt{\int W^2\mathrm{d}t}}}_{\mathrm{N}(0,1)}) \sim \mathrm{N}(0, \frac{\sigma^2}{\sigma_\eta^2}\left[2 + \frac{1}{T}\left(\int W^2\mathrm{d}t\right)^{-1}\right])\end{aligned}$$

- Hausman test statistic under the null: Obviously H1 and H2 would have the same limit distribution. Indeed for H2 we just drop the second term in the denominator (which is $O(T^{-1})$).

$$\begin{aligned}\mathrm{H1} &= \frac{(\sqrt{T}c)^2}{(T\hat{V}_D) + T^{-1}(T^2\hat{V}_L)} \\ &\Rightarrow \frac{(\sqrt{q}\int \mathrm{d}W\mathrm{d}^2U)^2}{2 + \frac{1}{T}(\int W^2\mathrm{d}t)^{-1}} \\ &\Rightarrow \frac{1}{2}(\underbrace{\int \mathrm{d}W\mathrm{d}^2U}_{\mathrm{N}(0,2)})^2 \sim \chi^2(1)\end{aligned}$$

- variance ratio test statistic under the null: For any bivariate case ($k = 1$)

$$\begin{aligned} \mathrm{J} &= \frac{T\hat{V}_L}{\hat{V}_D} = \frac{T^2\hat{V}_L}{T\hat{V}_D} \\ &\Rightarrow \frac{V_L}{V_D} = (2\int_0^1 W(t)^2 \mathrm{d}t)^{-1} = O(1) \end{aligned}$$

We will turn now to the asymptotic behaviour of our statistics under the alternative of no cointegration.

- $\hat{\beta}_L$ under the alternative:

$$\begin{aligned} (\hat{\beta}_L - \beta) &= \frac{T^{-2}\sum x_t u_t}{T^{-2}\sum x_t^2} = \frac{T^{-1}\sum(T^{-1/2}x_t)(T^{-1/2}u_t)}{T^{-1}\sum(T^{-1/2}x_t)^2} \\ &\Rightarrow \frac{\sigma_a}{\sigma_\eta}\frac{\int_0^1 W(t)A(t)\mathrm{d}t}{\int_0^1 W(t)^2\mathrm{d}t} = 0(1) \end{aligned}$$

Note then that $\hat{\beta}_L$ does not converge in probability to β.

- $\hat{\beta}_D$ under the alternative: Obviously $\hat{\beta}_D$ is efficient under the alternative because $u_t^\star = a_t$. Besides

$$\begin{aligned} \sqrt{T}(\hat{\beta}_D - \beta) &= \frac{T^{-1/2}\sum(\Delta x_t)u_t^\star}{T^{-1}\sum(\Delta x_t)^2} = \frac{T^{-1/2}\sum \eta_t a_t}{T^{-1}\sum \eta_t^2} \\ &\Rightarrow \frac{\sigma_a}{\sigma_\eta}\underbrace{\int_0^1 \mathrm{d}W(t)\mathrm{d}A(t)}_{\mathrm{N}(0,1)} \sim \mathrm{N}(0, V_D) \end{aligned}$$

where $V_D = \sigma_a^2/\sigma_\eta^2$

- $\hat{V}_L$ under the alternative:

$$\begin{aligned} T\hat{V}_L &= \frac{T^{-1}\hat{\sigma}^2}{T^{-2}\sum x_t^2} = \frac{T^{-2}\sum[u_t - (\hat{\beta}_L - \beta)x_t]^2}{T^{-2}\sum x_t^2} = \frac{T^{-2}\sum u_t^2}{T^{-2}\sum x_t^2} - (\hat{\beta}_L - \beta)^2 \\ &\Rightarrow \frac{\sigma_a^2}{\sigma_\eta^2}(\int W^2\mathrm{d}t)^{-2}\left[\int A^2\mathrm{d}t\int W^2\mathrm{d}t - \left(\int WA\mathrm{d}t\right)^2\right] = 0(1) \end{aligned}$$

- $\hat{V}_D$ under the alternative:

$$\begin{aligned} TV_D &= \frac{\hat{\sigma}_\star^2}{T^{-1}\sum(\Delta x_t)^2} = \frac{T^{-1}\sum \hat{a}_t^2}{T^{-1}\sum \eta_t^2} \\ &\Rightarrow \frac{\sigma_a^2}{\sigma_\eta^2} = V_D \end{aligned}$$

- $c = (\hat{\beta}_D - \hat{\beta}_L)$ under the alternative:

$$\begin{aligned} c &= (\hat{\beta}_D - \beta) - (\hat{\beta}_L - \beta) \\ &\Rightarrow -\frac{\sigma_a}{\sigma_\eta\sqrt{\int W^2\mathrm{d}t}}\underbrace{\frac{\int WA\mathrm{d}t}{\sqrt{\int W^2\mathrm{d}t}}}_{\mathrm{N}(0,1)} \sim \mathrm{N}(0, \frac{\sigma_a^2}{\sigma_\eta^2}[\int W^2\mathrm{d}t]^{-1}) \end{aligned}$$

- Hausman test statistic under the alternative:

$$
\begin{aligned}
T^{-1}\text{H1} &= \frac{c^2}{T\hat{V}_D + T\hat{V}_L} \\
&\Rightarrow \frac{\left(\frac{\int WA\mathrm{d}t}{\int W^2\mathrm{d}t}\right)^2}{1+\left[\frac{\int A^2\mathrm{d}t\int W^2\mathrm{d}t-(\int WA\mathrm{d}t)^2}{(\int W^2\mathrm{d}t)^2}\right]} \\
&= \left[\left(\frac{\int W^2\mathrm{d}t}{\int WA\mathrm{d}t}\right)^2 + \frac{\int A^2\mathrm{d}t\int W^2\mathrm{d}t}{(\int WA\mathrm{d}t)^2} - 1\right]^{-1} = O(1) \\
T^{-1}\text{H2} &= \frac{c^2}{T\hat{V}_D} \\
&\Rightarrow \left(\frac{\int WA\mathrm{d}t}{\int W^2\mathrm{d}t}\right)^2 = O(1)
\end{aligned}
$$

- variance ratio test statistic under the alternative:

$$
\begin{aligned}
T^{-1}\text{J} &= \frac{T\hat{V}_L}{T\hat{V}_D} \\
&\Rightarrow \left[\frac{\int A^2\mathrm{d}t\int W^2\mathrm{d}t-(\int WA\mathrm{d}t)^2}{(\int W^2\mathrm{d}t)^2}\right] = O(1)
\end{aligned}
$$

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Table 4: **Power of $H1$, $H2$ and J Statistics**

	Statistic $H1$			Statistic $H2$			Statistic J		
Sample size	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
T = 10	37.8	28.2	12.2	46.1	37.1	22.2	41.9	35.6	23.5
T = 20	53.7	45.5	32.0	61.6	55.2	43.2	59.2	52.3	40.9
T = 30	61.6	54.5	41.7	68.8	63.1	53.1	68.6	61.6	51.0
T = 40	66.3	59.7	48.7	73.4	68.2	59.3	74.4	67.8	57.1
T = 50	69.8	64.1	53.7	76.1	71.7	63.2	78.7	72.7	62.4
T = 100	77.9	73.9	65.9	82.8	79.7	73.4	90.2	85.8	77.0
T = 150	81.5	78.1	71.6	85.6	82.9	78.0	94.8	91.9	84.6
T = 200	84.1	81.4	75.7	87.6	85.5	81.2	96.8	94.5	88.5
T = 250	85.8	83.3	78.1	89.1	87.2	82.9	98.3	96.4	91.7
T = 500	90.1	88.4	84.7	92.6	91.1	88.2	99.7	99.3	97.8

Table 5: **Power of C Statistics**

	l_0			l_4			l_{12}		
Sample size	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
T = 10	44.8	32.9	12.3	23.3	3.7	0.0	1.7	0.0	0.0
T = 20	64.3	54.4	35.5	40.9	26.4	2.8	13.1	0.0	0.0
T = 30	75.6	65.6	48.4	50.4	38.4	15.8	24.4	4.8	0.0
T = 40	82.3	73.3	57.2	50.8	38.5	15.7	29.1	10.9	0.0
T = 50	86.4	78.8	63.2	56.1	44.7	23.6	32.2	15.2	0.0
T = 100	96.6	92.4	81.7	67.3	57.1	39.2	45.4	31.5	7.6
T = 150	98.6	96.7	89.9	77.4	67.7	50.8	52.8	41.0	18.7
T = 200	99.5	98.4	93.6	83.4	74.4	58.3	57.5	46.6	26.0
T = 250	99.7	99.1	96.0	84.1	75.5	59.5	61.2	50.8	31.2
T = 500	99.9	99.9	99.4	95.0	90.1	77.9	75.3	65.0	48.5

Table 6: **Power of C_μ Statistics**

	l_0			l_4			l_{12}		
Sample size	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
T = 10	56.1	42.1	20.8	47.0	27.7	0.0	-	-	-
T = 20	80.4	69.1	45.3	54.4	40.4	15.9	-	-	-
T = 30	90.8	83.0	62.4	66.3	51.0	28.5	-	-	-
T = 40	95.4	89.9	74.2	66.9	51.3	29.1	50.5	33.9	0.4
T = 50	97.2	93.5	81.1	73.6	58.8	35.9	51.3	35.8	4.0
T = 100	99.7	99.2	95.7	86.0	75.7	51.3	59.8	45.3	22.1
T = 150	99.9	99.8	98.8	93.1	86.4	67.2	69.8	54.2	32.4
T = 200	99.9	99.9	99.5	96.5	92.1	77.7	76.5	61.6	38.7
T = 250	100.0	100.0	99.8	96.8	92.8	79.6	80.9	68.1	43.3
T = 500	100.0	100.0	100.0	99.6	98.9	94.3	92.2	84.6	64.7

Table 1: **Critical Values for the $H1$ Statistic**

T	**k**	*Critical value* 0.25	0.50	0.75	0.90	0.95	0.99	**T**	**k**	*Critical value* 0.25	0.50	0.75	0.90	0.95	0.99
10	*1*	0.067	0.306	0.936	2.063	3.081	5.921	*100*	*1*	0.097	0.435	1.268	2.613	3.713	6.464
	2	0.282	0.709	1.547	2.842	3.964	7.203		*2*	0.521	1.271	2.553	4.302	5.623	8.794
	3	0.449	0.944	1.829	3.151	4.314	7.674		*3*	1.062	2.094	3.666	5.629	7.118	10.475
	4	0.555	1.072	1.957	3.291	4.518	8.018		*4*	1.646	2.878	4.662	6.831	8.422	11.955
20	*1*	0.081	0.367	1.099	2.327	3.399	6.304	*150*	*1*	0.099	0.442	1.284	2.639	3.751	6.520
	2	0.383	0.949	1.975	3.462	4.686	7.852		*2*	0.538	1.305	2.630	4.403	5.779	9.076
	3	0.697	1.404	2.570	4.181	5.468	8.768		*3*	1.104	2.183	3.812	5.856	7.363	10.816
	4	0.965	1.762	3.001	4.658	5.965	9.337		*4*	1.734	3.034	4.896	7.129	8.752	12.512
30	*1*	0.087	0.388	1.152	2.431	3.520	6.373	*200*	*1*	0.099	0.447	1.297	2.637	3.732	6.469
	2	0.433	1.067	2.180	3.758	5.030	8.160		*2*	0.549	1.326	2.672	4.470	5.855	9.084
	3	0.823	1.645	2.946	4.699	6.055	9.332		*3*	1.128	2.216	3.891	5.937	7.467	10.999
	4	1.181	2.118	3.549	5.373	6.791	10.072		*4*	1.772	3.101	5.002	7.287	8.924	12.619
40	*1*	0.091	0.406	1.182	2.469	3.585	6.387	*250*	*1*	0.099	0.446	1.298	2.670	3.739	6.466
	2	0.464	1.130	2.313	3.939	5.238	8.352		*2*	0.552	1.343	2.677	4.498	5.920	9.175
	3	0.900	1.783	3.167	4.979	6.335	9.776		*3*	1.149	2.254	3.915	6.033	7.569	11.078
	4	1.319	2.347	3.896	5.799	7.250	10.735		*4*	1.800	3.142	5.078	7.369	9.021	12.702
50	*1*	0.091	0.413	1.209	2.508	3.605	6.465	*500*	*1*	0.101	0.449	1.301	2.670	3.762	6.537
	2	0.480	1.172	2.397	4.032	5.297	8.380		*2*	0.567	1.356	2.728	4.566	5.938	9.169
	3	0.946	1.874	3.320	5.156	6.558	9.908		*3*	1.176	2.305	4.021	6.126	7.671	11.135
	4	1.411	2.510	4.125	6.127	7.635	11.159		*4*	1.857	3.244	5.214	7.561	9.198	13.007
								∞	*1*	0.102	0.455	1.323	2.706	3.841	6.635
									2	0.575	1.386	2.773	4.605	5.991	9.210
									3	1.213	2.366	4.108	6.251	7.815	11.345
									4	1.923	3.357	5.385	7.779	9.488	13.277

Table 2: **Critical Values for the $H2$ Statistic**

		Critical value								*Critical value*					
T	**k**	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*	*0.99*	**T**	**k**	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*	*0.99*
10	*1*	0.083	0.380	1.170	2.597	3.953	7.898	*100*	*1*	0.100	0.447	1.303	2.682	3.806	6.620
	2	0.370	0.930	2.032	3.791	5.366	10.353		*2*	0.545	1.328	2.663	4.492	5.882	9.188
	3	0.610	1.283	2.505	4.374	6.115	11.567		*3*	1.129	2.224	3.890	5.969	7.534	11.069
	4	0.766	1.483	2.752	4.785	6.749	13.144		*4*	1.777	3.102	5.023	7.349	9.058	12.868
20	*1*	0.092	0.413	1.236	2.620	3.823	7.102	*150*	*1*	0.101	0.450	1.307	2.682	3.816	6.626
	2	0.454	1.127	2.335	4.092	5.544	9.303		*2*	0.555	1.346	2.711	4.534	5.956	9.364
	3	0.859	1.728	3.157	5.132	6.709	10.772		*3*	1.151	2.276	3.979	6.108	7.667	11.263
	4	1.225	2.230	3.796	5.888	7.557	11.775		*4*	1.829	3.200	5.165	7.500	9.205	13.166
30	*1*	0.094	0.423	1.251	2.646	3.821	6.877	*200*	*1*	0.101	0.453	1.316	2.673	3.779	6.550
	2	0.492	1.205	2.467	4.240	5.667	9.229		*2*	0.561	1.358	2.733	4.570	5.984	9.285
	3	0.971	1.932	3.447	5.470	7.061	10.836		*3*	1.164	2.288	4.013	6.125	7.708	11.351
	4	1.427	2.557	4.261	6.429	8.129	12.122		*4*	1.846	3.231	5.212	7.579	9.288	13.122
40	*1*	0.097	0.432	1.262	2.633	3.820	6.811	*250*	*1*	0.100	0.450	1.313	2.701	3.779	6.532
	2	0.513	1.245	2.555	4.345	5.759	9.172		*2*	0.562	1.368	2.725	4.584	6.032	9.374
	3	1.023	2.030	3.597	5.642	7.175	10.979		*3*	1.180	2.314	4.016	6.187	7.758	11.369
	4	1.538	2.732	4.534	6.745	8.407	12.396		*4*	1.863	3.249	5.249	7.607	9.323	13.135
50	*1*	0.096	0.434	1.272	2.636	3.794	6.793	*500*	*1*	0.101	0.452	1.309	2.685	3.783	6.574
	2	0.521	1.272	2.595	4.376	5.736	9.079		*2*	0.572	1.370	2.754	4.607	6.001	9.236
	3	1.054	2.094	3.695	5.727	7.275	10.912		*3*	1.193	2.336	4.075	6.202	7.778	11.277
	4	1.610	2.861	4.690	6.969	8.655	12.605		*4*	1.889	3.300	5.308	7.684	9.356	13.208
								∞	*1*	0.102	0.455	1.323	2.706	3.841	6.635
									2	0.575	1.386	2.773	4.605	5.991	9.210
									3	1.213	2.366	4.108	6.251	7.815	11.345
									4	1.923	3.357	5.385	7.779	9.488	13.277

Table 1: **Critical Values for the J Statistic**

		Critical value								Critical value					
T	**m**	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*	*0.99*	**T**	**m**	*0.25*	*0.50*	*0.75*	*0.90*	*0.95*	*0.99*
10	*1*	0.746	1.629	3.479	6.304	8.708	15.649	*100*	*1*	0.779	1.711	3.624	6.451	8.693	14.083
	2	1.367	2.070	3.271	5.099	6.736	11.704		*2*	1.560	2.286	3.434	4.953	6.140	8.922
	3	1.843	2.563	3.733	5.484	7.118	12.472		*3*	2.302	2.983	3.903	5.031	5.869	7.822
	4	2.270	3.054	4.364	6.473	8.546	15.572		*4*	3.015	3.681	4.521	5.467	6.139	7.662
20	*1*	0.766	1.687	3.553	6.313	8.509	13.753	*150*	*1*	0.782	1.712	3.624	6.478	8.745	14.320
	2	1.477	2.191	3.329	4.912	6.162	9.344		*2*	1.564	2.287	3.428	4.979	6.169	9.065
	3	2.105	2.789	3.778	5.031	6.023	8.516		*3*	2.306	2.984	3.914	5.031	5.867	7.816
	4	2.677	3.376	4.330	5.535	6.464	8.768		*4*	3.032	3.694	4.527	5.467	6.125	7.677
30	*1*	0.771	1.693	3.586	6.343	8.586	13.857	*200*	*1*	0.782	1.716	3.615	6.490	8.788	14.381
	2	1.513	2.225	3.360	4.889	6.112	9.071		*2*	1.564	2.286	3.438	4.988	6.189	9.050
	3	2.185	2.867	3.810	5.010	5.914	8.117		*3*	2.312	2.992	3.915	5.033	5.881	7.848
	4	2.812	3.502	4.389	5.434	6.242	8.164		*4*	3.044	3.703	4.537	5.473	6.135	7.617
40	*1*	0.776	1.689	3.599	6.399	8.578	14.027	*250*	*1*	0.781	1.707	3.619	6.461	8.760	14.369
	2	1.527	2.248	3.382	4.923	6.128	9.102		*2*	1.567	2.288	3.434	4.985	6.185	9.081
	3	2.224	2.909	3.837	4.995	5.870	7.996		*3*	2.314	3.000	3.924	5.028	5.866	7.843
	4	2.884	3.560	4.432	5.449	6.180	7.959		*4*	3.050	3.708	4.537	5.487	6.148	7.638
50	*1*	0.775	1.693	3.595	6.426	8.694	13.976	*500*	*1*	0.782	1.719	3.630	6.491	8.772	14.405
	2	1.539	2.257	3.402	4.935	6.128	8.963		*2*	1.572	2.292	3.450	5.000	6.218	9.074
	3	2.247	2.931	3.859	5.002	5.846	7.920		*3*	2.323	3.006	3.923	5.029	5.867	7.803
	4	2.927	3.595	4.462	5.451	6.182	7.826		*4*	3.062	3.724	4.553	5.469	6.127	7.586