Semiparametric estimation in perturbed long memory series

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Perturbed Long Memory Series

$$z_t = y_t + u_t$$

- u_t is a weak dependent added noise with continuous, finite and positive spectral density (e.g. white noise).
- y_t is a stationary long memory signal with spectral density satisfying as $\lambda \to 0$

$$f_y(\lambda) = C\lambda^{-2d_0}(1 + O(\lambda^\alpha))$$

  - C a positive constant,
  - $\alpha \in [1, 2]$ ($\alpha = 2$ in standard fractional ARIMA),
  - the **memory parameter** d_0 satisfies $0 < d_0 < 1/2$.

Examples:

- Measurement errors in economic series.
- Different factors for long run and short run behaviour.
- Rational expectation with persistent ex ante variable.
- Long Memory in Stochastic Volatility (LMSV) models for financial asset returns (Harvey, 1998, Breidt, Crato and de Lima, 1998) satisfying:
  - Lack of temporal dependence in the returns (efficient market hypothesis).
  - High persistence or long memory in “proxies” of the volatility (squares or other powers of absolute values of returns).

LMSV

$$x_t = \sigma \sigma_t \varepsilon_t \qquad \sigma_t = \exp(y_t/2)$$

- σ is a positive constant.
- $\varepsilon_t \sim iid(0, 1)$.
- The **(log) volatility** process y_t is stationary long memory such that its spectral density satisfies as $\lambda \to 0$

$$f_y(\lambda) = C\lambda^{-2d_0}(1 + O(\lambda^\alpha))$$

- If ε_t and y_t are independent x_t is zero mean stationary with zero autocovariances.

LMSV

Taking logs of the squares

$$z_t = \log x_t^2 = \mu + y_t + u_t$$

- $\mu = \log \sigma^2 + E \log \varepsilon_t^2$
- $u_t = \log \varepsilon_t^2 - E \log \varepsilon_t^2$ is iid with zero mean and constant variance σ_u^2.

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z_t is a long memory signal plus noise model

Semiparametric estimation in perturbed LM

Whittle based methods:

- Gaussian semiparametric or local Whittle estimation (GSE): Kunsch (1987), Robinson (1995), Arteche (2004).
- Modified Gaussian semiparametric (MGSE): Hurvich, Moulines and Soulier (2005).

Log periodogram regression based methods:

- Log-periodogram regression estimation (LPE): Geweke and Porter-Hudak(1983), Robinson(1995), Deo and Hurvich(2001).
- Non linear log-periodogram (NLPE): Sun and Phillips (2003).
- Augmented log-periodogram regression estimation (ALPE).

Periodogram and spectral density

As $\lambda \to 0$

$$f_z(\lambda) = C\lambda^{-2d_0}(1+O(\lambda^\alpha))+f_u(\lambda) = C\lambda^{-2d_0}\left(1+\frac{f_u(0)}{C}\lambda^{2d_0}+O(\lambda^\alpha)\right)$$

Define the periodogram of z_t at Fourier frequency $\lambda_j = \frac{2\pi j}{n}$ as

$$I_{zj} = I_z(\lambda_j) = \frac{1}{2\pi n}\left|\sum_{t=1}^{n} z_t \exp(-i\lambda_j t)\right|^2$$

Periodogram and spectral density

Theorem 1: *Let* $d < 0.5$ *and define*

$$L_n(j) = E\left[\frac{I_{zj}}{C\lambda_j^{-2d_0}}\right].$$

Then, considering j *fixed:*

$$L_n(j) = A_{1n}(j) + A_{2n}(j) + o(n^{-2d_0})$$

where

$$\lim_{n\to\infty} A_{1n}(j) = \int_{-\infty}^{\infty} \psi_j(\lambda) \left|\frac{\lambda}{2\pi j}\right|^{-2d_0} d\lambda \;\; , \;\; \lim_{n\to\infty} n^{2d_0} A_{2n}(j) = \int_{-\infty}^{\infty} \psi_j(\lambda) \frac{f_u(0)}{C(2\pi j)^{-2d_0}} d\lambda$$

where

$$\psi_j(\lambda) = \frac{2}{\pi} \frac{\sin^2 \frac{\lambda}{2}}{(2\pi j - \lambda)^2}.$$

Gaussian semiparametric estimation (GSE)

$$\hat{d}_{GSE} = \arg\min R(d)$$

$$R(d) = \log\left(\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d} I_{zj}\right) - \frac{2d}{m}\sum_{j=1}^{m}\log\lambda_j$$

for the *bandwidth* m satisfying at least $\frac{1}{m} + \frac{m}{n} \longrightarrow 0$ as $n \longrightarrow \infty$.

- Bias of $\hat{d}_{GSE}$ is of order $O(\lambda_m^{2d_0})$.
- $\sqrt{m}(\hat{d}_{GSE} - d_0) \xrightarrow{d} N(0, \frac{1}{4})$ as long as $m = \kappa n^{\varsigma}$ for $\varsigma < 4d_0/(4d_0+1)$ (Arteche, 2004).

Modified GSE (MGSE) (Hurvich, Moulines and Soulier, 2005)

$$(\hat{d}_{MGSE}, \hat{\beta}_{MGSE}) = \arg\min_{\Delta \times \Theta} R(d, \beta)$$

$\Theta = [0, \Theta_1]$, $0 < \Theta_1 < \infty$, $\Delta = [\Delta_1, \Delta_2]$, $0 < \Delta_1 < \Delta_2 < 1/2$,

$$R(d, \beta) = \log\left(\frac{1}{m}\sum_{j=1}^{m} \frac{\lambda_j^{2d} I_{zj}}{1 + \beta\lambda_j^{2d}}\right) + \frac{1}{m}\sum_{j=1}^{m} \log\{\lambda_j^{-2d}(1 + \beta\lambda_j^{2d})\}$$

for $\beta_0 = f_u(0)/C$ the long run nsr.

- Bias of $\hat{d}_{MGSE}$ is of order $O(\lambda_m^{\alpha})$.
- $\sqrt{m}(\hat{d}_{MGSE} - d_0) \xrightarrow{d} N(0, \frac{1}{4}C_{d_0})$ for $C_{d_0} = 1 + \frac{1+4d_0}{4d_0^2}$ as long as $m = \kappa n^{\varsigma}$ for $\varsigma < 2\alpha/(2\alpha + 1)$.

Log-periodogram regression estimator (LPE)

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Taking the log of $f_z(\lambda)$

$$\log I_{zj} = a + d_0(-2\log\lambda_j) + \log\left(1 + \frac{f_u(\lambda)}{C}\lambda_j^{2d_0} + O(\lambda_j^{\alpha})\right) + U_{zj}$$

- $a = \log C - c$, $c = 0.577216...$ is Euler's constant.
- $U_{zj} = \log\left(\frac{I_{zj}}{f_z(\lambda_j)}\right) + c$.

Log-periodogram regression estimator (LPE)

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for $j = 1, 2, ..., m$.

- $a = \log C - c$, $c = 0.577216...$ is Euler's constant.
- $U_{zj} = \log\left(\frac{I_{zj}}{f_z(\lambda_j)}\right) + c$.
- m is the *bandwidth* satisfying at least $\frac{1}{m} + \frac{m}{n} \to 0$ as $n \to \infty$.

Log-periodogram regression estimator (LPE)

$\hat{d}_{LPE}$ (Robinson, 1995) is obtained by least squares to

$$\log I_{zj} = a + d(-2\log\lambda_j) + v_j \quad j = 1, 2, ..., m$$

- Bias of $\hat{d}_{LPE}$ is of order $O(\lambda_m^{2d_0})$.
- $\sqrt{m}(\hat{d}_{LPE} - d_0) \xrightarrow{d} N(0, \frac{\pi^2}{24})$ as long as $m = \kappa n^{\varsigma}$ for $\varsigma < 4d_0/(4d_0+1)$ (Deo and Hurvich, 2001).

$\hat{d}_{LPE}$ and $\hat{d}_{GSE}$ have a large downward bias caused by the added noise.

Non linear LPE (NLPE) (Sun and Phillips, 2003)

$$\begin{aligned}
\log I_{zj} &= a + d_0(-2\log\lambda_j) + \log\left(1 + \frac{f_u(\lambda_j)}{C}\lambda_j^{2d_0} + O(\lambda_j^{\alpha})\right) + U_{zj} \\
&= a + d_0(-2\log\lambda_j) + \log\left(1 + \frac{f_u(0)}{C}\lambda_j^{2d_0}\right) + O(\lambda_j^{\alpha}) + U_{zj} \\
&= a + d_0(-2\log\lambda_j) + \frac{f_u(0)}{C}\lambda_j^{2d_0} + O(\lambda_j^{\alpha^*}) + U_{zj}
\end{aligned}$$

where $\alpha^* = \min(4d_0, \alpha)$.

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&= a + d_0(-2\log\lambda_j) + \log\left(1 + \frac{f_u(0)}{C}\lambda_j^{2d_0}\right) + O(\lambda_j^{\alpha}) + U_{zj} \\
&= a + d_0(-2\log\lambda_j) + \frac{f_u(0)}{C}\lambda_j^{2d_0} + O(\lambda_j^{\alpha^*}) + U_{zj}
\end{aligned}$$

where $\alpha^* = \min(4d_0, \alpha)$.

Non linear LPE (NLPE) (Sun and Phillips, 2003)

$$(\hat{d}_{NLPE}, \hat{\beta}_{NLPE}) = \arg\min_{\Delta\times\Theta} \sum_{j=1}^{m} (\log^* I_{zj} + d(2\log\lambda_j)^* - \beta(\lambda_j^{2d})^*)^2$$

where for a general ξ_t, $\xi_t^* = \xi_t - \bar{\xi}$ for $\bar{\xi} = \sum \xi_t / n$.

- Bias of $\hat{d}_{NLPE}$ is of order $O(\lambda_m^{\alpha^*})$.
- $\sqrt{m}(\hat{d}_{NLPE} - d_0) \xrightarrow{d} N(0, \frac{\pi^2}{24} C_{d_0})$ as long as $m = \kappa n^{\varsigma}$ for $\varsigma < 2\alpha^*/(2\alpha^* + 1)$.

For $\alpha > 4d_0$ (e.g. in ARFIMA $\alpha = 2$) $\alpha^* = 4d_0$.

Augmented log-periodogram estimation (ALPE)

$$
\begin{aligned}
\log I_{zj} &= a + d_0(-2\log\lambda_j) + \log\left(1 + \frac{f_u(\lambda_j)}{C}\lambda_j^{2d_0} + O(\lambda_j^{\alpha})\right) + U_{zj} \\
&= a + d_0(-2\log\lambda_j) + \log\left(1 + \frac{f_u(0)}{C}\lambda_j^{2d_0}\right) + O(\lambda_j^{\alpha}) + U_{zj} \\
&= a + d_0(-2\log\lambda_j) + \frac{f_u(0)}{C}\lambda_j^{2d_0} + O(\lambda_j^{\alpha^*}) + U_{zj}
\end{aligned}
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&= a + d_0(-2\log\lambda_j) + \frac{f_u(0)}{C}\lambda_j^{2d_0} + O(\lambda_j^{\alpha^*}) + U_{zj}
\end{aligned}$$

Augmented log-periodogram estimation (ALPE)

$$(\hat{d}_{ALPE}, \hat{\beta}_{ALPE}) = \arg\min_{\Delta\times\Theta} Q(d,\beta)$$

under the constraint $\beta \geq 0$, where

$$Q(d,\beta) = \sum_{j=1}^{m} (\log^* I_{zj} + d(2\log\lambda_j)^* - \log^*(1+\beta\lambda_j^{2d}))^2$$

- Bias of $\hat{d}_{ALPE}$ is of order $O(\lambda_m^{\alpha})$.
- $\sqrt{m}(\hat{d}_{ALPE} - d_0) \xrightarrow{d} N(0, \frac{\pi^2}{24}C_{d_0})$ as long as $m = \kappa n^{\varsigma}$ for $\varsigma < 2\alpha/(2\alpha+1)$.

Assumptions

- **B.1**: y_t and u_t are independent Gaussian processes.
- **B.2**: When $\sigma_u^2 > 0$, $f_u(\lambda)$ is continuous on $[-\pi, \pi]$, bounded above and away from zero with bounded first derivative in a neighbourhood of zero.
- **B.3**: The spectral density of y_t satisfies as $\lambda \to 0$,

  $$f_y(\lambda) = C\lambda^{-2d_0}(1 + G\lambda^{\alpha} + O(\lambda^{\alpha+\iota}))$$

  for some $\iota > 0$, finite positive C, finite G, $0 < d_0 < 0.5$ and $\alpha \in (4d_0, 2] \bigcap [1, 2]$.
- **B.4**: As $n \to \infty$, for some positive constant K,

  $$\frac{m^{2\alpha+1}}{n^{2\alpha}} \to K$$

Main result: Properties of $\hat{d}_{ALPE}$

When $var(u_t) > 0$, $\hat{d}_{ALPE}$ is asymptotically normal and

- $ABias(\hat{d}_{ALPE}) = K_0 \left(\frac{m}{n}\right)^{\alpha}, \quad K_0 = \frac{(2\pi)^{\alpha}\alpha(2d_0+1)(\alpha-2d_0)G}{4d_0(1+\alpha)^2(2d_0+\alpha+1)}$
- $AVar(\hat{d}_{ALPE}) = \frac{\pi^2}{24m}C_{d_0}$ where $C_{d_0} = 1 + \frac{1+4d_0}{4d_0^2}$
- $AMSE(\hat{d}_{ALPE}) = \frac{\pi^2}{24m}C_{d_0} + \left(\frac{m}{n}\right)^{2\alpha} K_0^2$.
- The optimal bandwidth, in an AMSE sense, is

$$m_{opt} = \left(\frac{\pi^2 C_{d_0}}{48\alpha K_0^2}\right)^{\frac{1}{2\alpha+1}} n^{\frac{2\alpha}{2\alpha+1}}.$$

Comparing estimators

	GSE	MGSE	LPE	NLPE	ALPE
$ABias$	$O\left(\left(\frac{m}{n}\right)^{2d_0}\right)$	$O\left(\left(\frac{m}{n}\right)^{\alpha}\right))$	$O\left(\left(\frac{m}{n}\right)^{2d_0}\right)$	$O\left(\left(\frac{m}{n}\right)^{4d_0}\right)$	$O\left(\left(\frac{m}{n}\right)^{\alpha}\right)$
m_{opt}	$O(n^{\frac{4d_0}{4d_0+1}})$	$O(n^{\frac{2\alpha}{2\alpha+1}})$	$O(n^{\frac{4d_0}{4d_0+1}})$	$O(n^{\frac{8d_0}{8d_0+1}})$	$O(n^{\frac{2\alpha}{2\alpha+1}})$
$AMSE(m_{opt})$	$O(n^{-\frac{4d_0}{4d_0+1}})$	$O(n^{-\frac{2\alpha}{2\alpha+1}})$	$O(n^{-\frac{4d_0}{4d_0+1}})$	$O(n^{-\frac{8d_0}{8d_0+1}})$	$O(n^{-\frac{2\alpha}{2\alpha+1}}$

Finite sample behaviour

- $z_t = y_t + u_t$.
- $u_t = \log \varepsilon_t^2$, for $\varepsilon_t \sim N(0,1)$.
- $(1-L)^{d_0} y_t = w_t$ with $w_t \sim N(0, \sigma_w^2)$,
  $d_0 = 0.2, 0.45, 0.8$.
- $\sigma_w^2 = 0.5, 0.1$, correspond to long run noise to signal ratios $f_u(0)/f_w(0) = \pi^2$, $5\pi^2$.
- w_t and ε_t independent.
- $n = 1024, 4096, 8192$.
- $m = n^{0.4}, n^{0.6}, n^{0.8}, m^{opt}$, 1000 replications.

“Optimal” bandwidths

Table 1: “Optimal” bandwidths

		$\sigma_w^2 = 0.5$					$\sigma_w^2 = 0.1$				
n	d_0	LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE
	0.2	6	5	12	511	511	5	5	5	511	511
1024	0.45	13	11	29	511	502	5	5	7	511	502
	0.8	27	24	53	511	511	12	11	22	511	511
	0.2	12	9	29	1895	1715	5	5	5	1895	1715
4096	0.45	32	27	87	1681	1522	10	8	21	1681	1522
	0.8	79	70	177	2047	1936	36	32	74	2047	1936
	0.2	16	12	45	3299	2987	5	5	5	3299	2987
8192	0.45	51	42	149	2927	2650	16	13	36	2927	2650
	0.8	134	119	323	3723	3370	62	55	135	3723	3370

Bias and MSE

Table 2: Bias and MSE

			$m = n^{0.4}$					$m = n^{0.6}$					$m = n^{0.8}$					$m = m_{est}^{opt}$				
n	σ_w^2		LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE
			$d_0 = 0.2$																			
1024	0.5	Bias	-0.075	-0.101	0.110	0.111	0.042	-0.132	-0.144	-0.013	0.078	0.037	-0.163	-0.167	-0.080	0.049	0.028	-0.005	-0.015	0.149	0.043	0.022
		MSE	0.025	0.023	0.073	0.128	0.097	0.022	0.023	0.019	0.109	0.092	0.027	0.028	0.014	0.091	0.081	0.057	0.058	0.102	0.079	0.067
	0.1	Bias	-0.103	-0.127	0.090	0.054	-0.001	-0.152	-0.164	-0.041	-0.004	-0.037	-0.175	-0.180	-0.102	-0.032	-0.058	0.018	-0.029	0.288	-0.032	-0.061
		MSE	0.025	0.025	0.071	0.110	0.083	0.026	0.028	0.020	0.082	0.072	0.031	0.033	0.017	0.070	0.064	0.075	0.054	0.228	0.070	0.063
4096	0.5	Bias	-0.093	-0.106	0.076	0.105	0.053	-0.137	-0.143	-0.047	0.069	0.033	-0.162	-0.164	-0.093	0.055	0.033	-0.046	-0.062	0.062	0.031	0.016
		MSE	0.019	0.019	0.050	0.093	0.069	0.021	0.022	0.013	0.072	0.058	0.027	0.027	0.013	0.057	0.044	0.031	0.032	0.043	0.041	0.030
	0.1	Bias	-0.125	-0.142	0.039	0.016	-0.027	-0.165	-0.170	-0.076	-0.017	-0.041	-0.182	-0.184	-0.125	-0.023	-0.036	0.015	-0.026	0.199	-0.054	-0.069
		MSE	0.023	0.025	0.042	0.069	0.057	0.029	0.030	0.016	0.062	0.052	0.033	0.034	0.019	0.055	0.051	0.076	0.059	0.150	0.047	0.043
8192	0.5	Bias	-0.091	-0.105	0.060	0.098	0.056	-0.136	-0.139	-0.055	0.066	0.037	-0.162	-0.162	-0.092	0.037	0.006	-0.057	-0.064	0.033	0.022	-0.001
		MSE	0.017	0.017	0.037	0.075	0.059	0.020	0.020	0.012	0.060	0.045	0.026	0.026	0.012	0.034	0.022	0.024	0.026	0.027	0.025	0.015
	0.1	Bias	-0.132	-0.146	0.009	0.003	-0.034	-0.169	-0.173	-0.090	-0.012	-0.045	-0.184	-0.185	-0.138	-0.045	-0.050	0.026	-0.014	0.183	-0.043	-0.041
		MSE	0.023	0.025	0.029	0.057	0.047	0.029	0.031	0.016	0.055	0.046	0.034	0.034	0.022	0.045	0.042	0.079	0.063	0.131	0.041	0.037
			$d_0 = 0.45$																			
1024	0.5	Bias	-0.107	-0.132	0.081	0.162	0.091	-0.222	-0.225	-0.065	0.111	0.042	-0.321	-0.315	-0.173	0.062	0.012	-0.105	-0.123	0.000	0.049	0.019
		MSE	0.049	0.047	0.075	0.127	0.103	0.057	0.056	0.024	0.095	0.072	0.105	0.100	0.039	0.062	0.042	0.056	0.059	0.041	0.048	0.030
	0.1	Bias	-0.247	-0.276	-0.031	0.033	-0.045	-0.344	-0.354	-0.192	-0.007	-0.045	-0.400	-0.401	-0.286	-0.039	-0.061	-0.096	-0.141	0.122	-0.028	-0.032
		MSE	0.088	0.096	0.071	0.131	0.118	0.124	0.129	0.060	0.129	0.119	0.161	0.162	0.090	0.117	0.107	0.121	0.114	0.140	0.116	0.103
4096	0.5	Bias	-0.054	-0.073	0.081	0.130	0.071	-0.165	-0.166	-0.042	0.045	0.008	-0.294	-0.281	-0.160	0.016	0.005	-0.069	-0.076	-0.011	0.017	0.009
		MSE	0.025	0.021	0.049	0.069	0.051	0.031	0.030	0.012	0.031	0.021	0.087	0.079	0.034	0.012	0.007	0.022	0.020	0.014	0.008	0.005
	0.1	Bias	-0.181	-0.198	0.002	0.070	0.017	-0.310	-0.308	-0.157	0.051	0.009	-0.392	-0.385	-0.258	0.033	0.007	-0.094	-0.122	0.023	0.028	0.007
		MSE	0.051	0.053	0.045	0.072	0.063	0.100	0.097	0.037	0.063	0.051	0.154	0.148	0.070	0.048	0.033	0.074	0.079	0.059	0.040	0.027
8192	0.5	Bias	-0.039	-0.050	0.087	0.127	0.078	-0.141	-0.140	-0.033	0.023	0.003	-0.282	-0.267	-0.127	0.012	0.004	-0.053	-0.055	-0.012	0.010	0.005
		MSE	0.018	0.013	0.038	0.053	0.037	0.022	0.021	0.008	0.017	0.011	0.080	0.072	0.018	0.006	0.003	0.012	0.011	0.008	0.004	0.002
	0.1	Bias	-0.154	-0.161	-0.001	0.058	0.019	-0.284	-0.280	-0.133	0.028	0.002	-0.383	-0.372	-0.240	0.032	0.010	-0.085	-0.097	0.008	0.027	0.010
		MSE	0.039	0.037	0.035	0.051	0.043	0.083	0.080	0.026	0.036	0.028	0.147	0.139	0.059	0.021	0.013	0.047	0.048	0.033	0.018	0.011
			$d_0 = 0.8$																			
1024	0.5	Bias	-0.019	-0.031	0.057	0.070	0.043	-0.141	-0.148	-0.007	0.032	0.007	-0.410	-0.372	-0.174	0.041	0.024	-0.037	-0.039	0.010	0.039	0.026
		MSE	0.035	0.028	0.037	0.038	0.033	0.030	0.029	0.017	0.023	0.019	0.171	0.141	0.035	0.016	0.011	0.024	0.018	0.018	0.014	0.010
	0.1	Bias	-0.110	-0.138	0.027	0.050	0.001	-0.344	-0.339	-0.124	0.021	-0.009	-0.589	-0.541	-0.356	0.026	0.009	-0.082	-0.095	0.001	0.032	0.014
		MSE	0.055	0.053	0.042	0.045	0.043	0.129	0.123	0.035	0.035	0.033	0.349	0.296	0.137	0.026	0.020	0.057	0.053	0.035	0.025	0.020
4096	0.5	Bias	0.016	0.007	0.074	0.083	0.062	-0.060	-0.065	0.023	0.041	0.021	-0.352	-0.307	-0.132	0.030	0.021	-0.010	-0.012	0.011	0.028	0.021
		MSE	0.021	0.015	0.026	0.027	0.021	0.008	0.007	0.008	0.011	0.007	0.125	0.096	0.019	0.005	0.003	0.008	0.006	0.007	0.004	0.003
	0.1	Bias	-0.015	-0.032	0.055	0.063	0.036	-0.216	-0.210	-0.044	0.035	0.021	-0.538	-0.464	-0.294	0.032	0.019	-0.029	-0.035	0.007	0.031	0.024
		MSE	0.022	0.016	0.026	0.028	0.021	0.052	0.048	0.010	0.015	0.011	0.290	0.217	0.088	0.010	0.006	0.018	0.014	0.014	0.008	0.006
8192	0.5	Bias	0.025	0.014	0.078	0.083	0.062	-0.036	-0.039	0.025	0.035	0.022	-0.321	-0.279	-0.112	0.025	0.020	-0.006	-0.008	0.012	0.025	0.019
		MSE	0.014	0.009	0.020	0.021	0.015	0.004	0.003	0.006	0.007	0.005	0.104	0.079	0.014	0.003	0.002	0.004	0.003	0.004	0.003	0.002
	0.1	Bias	0.012	-0.003	0.071	0.075	0.051	-0.164	-0.159	-0.020	0.031	0.021	-0.514	-0.433	-0.271	0.027	0.019	-0.012	-0.015	0.016	0.029	0.022
		MSE	0.015	0.011	0.021	0.021	0.016	0.030	0.028	0.006	0.010	0.006	0.265	0.188	0.075	0.006	0.003	0.010	0.008	0.009	0.005	0.003

90% CI d=0.2

Table 3: 90% Confidence Intervals ($d_0 = 0.2$)

		$m = n^{0.4}$					$m = n^{0.6}$					$m = n^{0.8}$					$m = m_{est}^{opt}$				
σ_w^2		LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE
											$n = 1024$										
0.5	Prob.A	0.977	0.982	0.906	0.764	0.798	0.396	0.208	0.983	0.739	0.768	0.002	0.000	0.998	0.709	0.728	0.930	0.906	0.882	0.673	0.693
	Mean.A	0.527	0.411	6.508	11.13	9.769	0.264	0.206	3.225	5.047	4.048	0.132	0.103	1.713	2.412	1.885	0.861	0.736	7.334	1.631	1.241
	Med.A	0.527	0.411	1.451	2.168	3.050	0.264	0.206	1.013	1.283	1.463	0.132	0.103	0.748	0.898	0.888	0.861	0.736	1.592	0.513	0.470
	Prob.H	0.994	0.993	0.924	0.985	1.000	0.476	0.254	0.947	0.964	1.000	0.003	0.000	0.886	0.966	0.998	0.986	0.980	0.904	0.974	0.999
	Mean.H	0.690	0.538	1.878	82783.5	63447.0	0.294	0.229	0.817	19971.3	14448.7	0.137	0.107	0.477	5791.7	3820.4	1.424	1.294	2.087	3093.3	1988.4
	Med.H	0.690	0.538	1.319	6.082	11.56	0.294	0.229	0.645	3.199	4.433	0.137	0.107	0.379	2.126	2.360	1.424	1.294	1.576	1.582	1.737
0.1	Prob.A	0.988	0.993	0.902	0.800	0.831	0.287	0.084	0.988	0.824	0.829	0.002	0.000	1.000	0.801	0.832	0.915	0.909	0.772	0.785	0.817
	Mean.A	0.527	0.411	7.322	13.498	11.53	0.264	0.206	4.144	6.946	5.840	0.132	0.103	2.455	3.723	3.183	0.944	0.736	9.905	2.724	2.294
	Med.A	0.527	0.411	1.581	5.116	16.83	0.264	0.206	1.241	4.345	5.406	0.132	0.103	0.972	3.264	5.243	0.944	0.736	1.937	3.372	3.505
	Prob.H	0.998	0.995	0.899	0.984	1.000	0.340	0.119	0.962	0.980	1.000	0.002	0.000	0.843	0.985	1.000	1.000	0.985	0.907	0.984	1.000
	Mean.H	0.690	0.538	1.726	101888.4	77328.3	0.294	0.229	0.822	28820.7	23462.4	0.137	0.107	0.476	10282.9	9016.2	1.660	1.294	5.916	6794.8	5265.9
	Med.H	0.690	0.538	1.306	14.901	51.87	0.294	0.229	0.675	8.338	13.805	0.137	0.107	0.383	4.899	10.456	1.660	1.294	3.841	4.488	5.050
											$n = 4096$										
0.5	Prob.A	0.991	0.601	0.900	0.696	0.742	0.157	0.034	0.990	0.686	0.718	0.000	0.000	1.000	0.629	0.642	0.951	0.950	0.930	0.641	0.643
	Mean.A	0.406	0.317	4.554	7.160	5.550	0.174	0.136	1.954	2.350	1.805	0.076	0.059	0.803	0.815	0.541	0.609	0.548	4.380	0.498	0.359
	Med.A	0.406	0.317	1.231	1.399	1.412	0.174	0.136	0.786	0.746	0.748	0.076	0.059	0.432	0.259	0.222	0.609	0.548	1.175	0.171	0.142
	Prob.H	0.995	0.997	0.889	0.956	1.000	0.185	0.044	0.935	0.955	1.000	0.000	0.000	0.806	0.940	0.990	0.986	0.985	0.914	0.962	0.984
	Mean.H	0.492	0.383	1.392	38375.2	25984.9	0.185	0.144	0.630	5768.3	3845.2	0.077	0.060	0.296	776.0	276.6	0.842	0.809	1.258	304.06	94.355
	Med.H	0.492	0.383	1.008	3.396	4.959	0.185	0.144	0.474	1.751	1.970	0.077	0.060	0.247	1.037	0.917	0.842	0.809	0.982	0.763	0.657
0.1	Prob.A	0.996	0.405	0.933	0.802	0.818	0.037	0.007	0.998	0.778	0.801	0.000	0.000	1.000	0.733	0.753	0.908	0.903	0.969	0.770	0.787
	Mean.A	0.406	0.317	5.878	10.11	8.640	0.174	0.136	2.847	4.403	3.515	0.076	0.059	1.432	1.966	1.562	0.944	0.736	10.31	1.389	1.179
	Med.A	0.406	0.317	1.389	3.597	6.523	0.174	0.136	0.988	2.345	2.199	0.076	0.059	0.674	1.533	1.403	0.944	0.736	2.137	1.519	1.483
	Prob.H	0.999	1.000	0.911	0.974	1.000	0.047	0.007	0.927	0.983	1.000	0.000	0.000	0.744	0.978	1.000	1.000	0.974	0.896	0.983	1.000
	Mean.H	0.492	0.383	1.295	57661.7	45030.0	0.185	0.144	0.591	13555.5	11057.0	0.077	0.060	0.332	3942.6	2938.7	1.660	1.294	8.354	2416.8	1967.8
	Med.H	0.492	0.383	0.993	8.628	19.74	0.185	0.144	0.466	3.977	4.913	0.077	0.060	0.267	2.360	2.731	1.660	1.294	4.052	2.211	2.557
											$n = 8192$										
0.5	Prob.A	0.694	0.560	0.939	0.705	0.725	0.062	0.005	0.995	0.666	0.698	0.000	0.000	1.000	0.616	0.676	0.967	0.953	0.949	0.614	0.508
	Mean.A	0.352	0.274	3.721	5.425	4.226	0.142	0.110	1.425	1.530	1.023	0.057	0.045	0.495	0.426	0.306	0.527	0.475	2.987	0.271	0.198
	Med.A	0.352	0.274	1.071	1.110	1.068	0.142	0.110	0.658	0.571	0.484	0.057	0.045	0.310	0.190	0.160	0.527	0.475	1.020	0.124	0.106
	Prob.H	0.998	0.651	0.908	0.950	1.000	0.071	0.008	0.919	0.941	0.996	0.000	0.000	0.823	0.952	0.980	0.985	0.980	0.934	0.955	0.969
	Mean.H	0.412	0.321	1.157	24013.3	16070.3	0.148	0.116	0.579	2629.3	1019.3	0.058	0.045	0.236	79.33	15.132	0.690	0.656	1.073	30.482	10.043
	Med.H	0.412	0.321	0.887	2.474	3.687	0.148	0.116	0.406	1.271	1.262	0.058	0.045	0.194	0.716	0.575	0.690	0.656	0.834	0.524	0.436
0.1	Prob.A	0.525	0.331	0.970	0.804	0.825	0.007	0.000	0.999	0.740	0.786	0.000	0.000	1.000	0.748	0.750	0.899	0.894	0.965	0.717	0.652
	Mean.A	0.352	0.274	5.083	8.360	6.906	0.142	0.110	2.332	3.328	2.739	0.057	0.045	1.267	1.534	1.177	0.944	0.736	9.559	0.970	0.736
	Med.A	0.352	0.274	1.300	2.773	3.086	0.142	0.110	0.891	1.654	1.811	0.057	0.045	0.662	1.319	1.081	0.944	0.736	2.077	0.933	0.369
	Prob.H	1.000	0.437	0.946	0.985	1.000	0.008	0.000	0.901	0.980	1.000	0.000	0.000	0.581	0.979	0.999	1.000	0.970	0.929	0.973	0.998
	Mean.H	0.412	0.321	1.234	41115.5	31034.2	0.148	0.116	0.540	8644.6	6411.0	0.058	0.045	0.295	2580.1	1860.5	1.660	1.294	10.06	1362.8	925.67
	Med.H	0.412	0.321	0.875	5.970	8.663	0.148	0.116	0.415	2.843	3.701	0.058	0.045	0.233	2.124	2.007	1.660	1.294	4.424	1.586	1.536

Prob.A, Mean.A, Med.A denote coverage percentages, mean lengths and median lengths of the nominal 90% confidence intervals with the asymptotic expression for standard errors with estimated parameters. Prob.H, Mean.H, Med.H denote coverage percentages, mean lengths and median lengths of the nominal 90% confidence intervals with the finite sample Hessian based approximation of the standard errors with estimated parameters.

90% CI d=0.45

Table 4: 90% Confidence Intervals ($d_0 = 0.45$)

		$m = n^{0.4}$					$m = n^{0.6}$					$m = n^{0.8}$					$m = m_{est}^{opt}$				
σ_w^2		LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE
												$n = 1024$									
0.5	Prob.A	0.747	0.625	0.881	0.677	0.710	0.153	0.037	0.988	0.632	0.717	0.000	0.000	0.724	0.515	0.443	0.745	0.653	0.944	0.393	0.382
	Mean.A	0.527	0.411	2.236	2.658	2.002	0.264	0.206	0.721	0.685	0.540	0.132	0.103	0.550	0.324	0.257	0.585	0.496	1.324	0.225	0.171
	Med.A	0.527	0.411	1.022	0.940	0.808	0.264	0.206	0.598	0.510	0.442	0.132	0.103	0.358	0.266	0.217	0.585	0.496	0.828	0.190	0.153
	Prob.H	0.864	0.776	0.922	0.995	1.000	0.192	0.055	0.943	0.984	1.000	0.000	0.000	0.415	0.977	0.939	0.882	0.807	0.923	0.956	0.934
	Mean.H	0.690	0.538	1.824	12658.2	6947.8	0.294	0.229	0.637	342.48	73.813	0.137	0.107	0.397	26.769	21.098	0.796	0.698	1.139	17.577	0.563
	Med.H	0.690	0.538	1.399	2.304	2.540	0.294	0.229	0.572	1.285	1.046	0.137	0.107	0.278	0.776	0.616	0.796	0.698	0.963	0.630	0.507
0.1	Prob.A	0.491	0.306	0.948	0.767	0.765	0.005	0.000	0.997	0.658	0.678	0.000	0.000	0.513	0.612	0.481	0.906	0.550	1.000	0.462	0.384
	Mean.A	0.527	0.411	3.821	6.019	5.017	0.264	0.206	1.933	2.826	1.965	0.132	0.103	1.086	1.486	1.021	0.944	0.736	5.956	1.066	0.721
	Med.A	0.527	0.411	1.174	1.125	1.080	0.264	0.206	0.780	0.638	0.549	0.132	0.103	0.511	0.309	0.254	0.944	0.736	1.458	0.217	0.165
	Prob.H	0.647	0.440	0.956	0.989	1.000	0.010	0.000	0.874	0.990	1.000	0.000	0.000	0.090	0.985	0.999	1.000	1.000	0.999	0.986	0.985
	Mean.H	0.690	0.538	1.835	39960.7	27831.6	0.294	0.229	0.735	9319.8	5207.6	0.137	0.107	0.424	3333.6	1786.2	1.660	1.294	3.457	2201.4	1267.4
	Med.H	0.690	0.538	1.363	3.093	4.597	0.294	0.229	0.613	1.960	2.468	0.137	0.107	0.338	1.582	1.618	1.660	1.294	2.593	1.408	1.492
												$n = 4096$									
0.5	Prob.A	0.805	0.750	0.884	0.745	0.769	0.085	0.015	0.985	0.742	0.748	0.000	0.000	0.219	0.559	0.534	0.781	0.724	0.966	0.486	0.494
	Mean.A	0.406	0.317	1.037	1.106	0.756	0.174	0.136	0.404	0.376	0.301	0.076	0.059	0.482	0.162	0.127	0.373	0.317	0.524	0.109	0.089
	Med.A	0.406	0.317	0.793	0.755	0.634	0.174	0.136	0.384	0.351	0.287	0.076	0.059	0.197	0.157	0.124	0.373	0.317	0.487	0.107	0.088
	Prob.H	0.886	0.822	0.899	0.927	0.997	0.110	0.021	0.956	0.960	0.959	0.000	0.000	0.110	0.905	0.896	0.864	0.813	0.944	0.903	0.893
	Mean.H	0.492	0.383	1.317	1415.0	263.90	0.185	0.144	0.395	0.683	0.499	0.077	0.060	0.297	0.353	0.272	0.443	0.383	0.589	0.291	0.230
	Med.H	0.492	0.383	1.072	1.415	1.290	0.185	0.144	0.364	0.620	0.484	0.077	0.060	0.148	0.345	0.269	0.443	0.383	0.509	0.283	0.227
0.1	Prob.A	0.564	0.373	0.945	0.811	0.756	0.000	0.000	1.000	0.606	0.558	0.000	0.000	0.028	0.324	0.285	0.718	0.631	0.918	0.252	0.234
	Mean.A	0.406	0.317	1.528	1.890	1.309	0.174	0.136	0.670	0.589	0.399	0.076	0.059	0.316	0.234	0.149	0.667	0.582	1.848	0.145	0.102
	Med.A	0.406	0.317	0.851	0.773	0.661	0.174	0.136	0.457	0.344	0.284	0.076	0.059	0.266	0.154	0.125	0.667	0.582	0.948	0.107	0.089
	Prob.H	0.688	0.483	0.928	0.958	1.000	0.000	0.000	0.850	0.940	0.997	0.000	0.000	0.006	0.931	0.944	0.984	0.976	0.914	0.906	0.917
	Mean.H	0.492	0.383	1.321	6247.8	2912.0	0.185	0.144	0.464	499.19	146.53	0.077	0.060	0.205	79.495	0.619	0.959	0.884	1.508	13.249	0.535
	Med.H	0.492	0.383	0.990	1.637	1.870	0.185	0.144	0.376	0.937	0.874	0.077	0.060	0.177	0.678	0.557	0.959	0.884	1.143	0.588	0.483
												$n = 8192$									
0.5	Prob.A	0.815	0.773	0.887	0.749	0.767	0.051	0.010	0.976	0.793	0.759	0.000	0.000	0.037	0.578	0.578	0.812	0.774	0.983	0.492	0.520
	Mean.A	0.352	0.274	0.790	0.769	0.574	0.142	0.110	0.319	0.303	0.239	0.057	0.045	0.166	0.121	0.095	0.295	0.254	0.380	0.082	0.067
	Med.A	0.352	0.274	0.674	0.650	0.546	0.142	0.110	0.311	0.294	0.233	0.057	0.045	0.145	0.119	0.094	0.295	0.254	0.368	0.081	0.067
	Prob.H	0.883	0.832	0.893	0.918	0.991	0.064	0.011	0.958	0.950	0.939	0.000	0.000	0.011	0.903	0.894	0.872	0.843	0.968	0.910	0.906
	Mean.H	0.412	0.321	1.032	105.44	1.065	0.148	0.116	0.310	0.478	0.353	0.058	0.045	0.119	0.241	0.187	0.335	0.293	0.416	0.199	0.159
	Med.H	0.412	0.321	0.855	1.111	0.989	0.148	0.116	0.290	0.449	0.349	0.058	0.045	0.108	0.238	0.186	0.335	0.293	0.363	0.197	0.158
0.1	Prob.A	0.587	0.417	0.947	0.857	0.781	0.000	0.000	0.931	0.653	0.523	0.000	0.000	0.001	0.375	0.333	0.762	0.683	0.940	0.271	0.265
	Mean.A	0.352	0.274	1.160	1.191	0.815	0.142	0.110	0.405	0.331	0.260	0.057	0.045	0.202	0.125	0.097	0.527	0.456	0.984	0.086	0.069
	Med.A	0.352	0.274	0.738	0.680	0.580	0.142	0.110	0.361	0.290	0.231	0.057	0.045	0.192	0.117	0.094	0.527	0.456	0.734	0.080	0.067
	Prob.H	0.668	0.506	0.933	0.956	1.000	0.000	0.000	0.710	0.943	0.945	0.000	0.000	0.000	0.932	0.920	0.866	0.822	0.932	0.919	0.899
	Mean.H	0.412	0.321	1.119	1865.9	654.26	0.148	0.116	0.336	0.743	0.589	0.058	0.045	0.133	0.487	0.367	0.690	0.620	1.131	6.468	0.330
	Med.H	0.412	0.321	0.856	1.304	1.276	0.148	0.116	0.292	0.682	0.559	0.058	0.045	0.126	0.443	0.351	0.690	0.620	0.834	0.396	0.317

Prob.A, Mean.A, Med.A denote coverage percentages, mean lengths and median lengths of the nominal 90% confidence intervals with the asymptotic expression for standard errors with estimated parameters. Prob.H, Mean.H, Med.H denote coverage percentages, mean lengths and median lengths of the nominal 90% confidence intervals with the finite sample Hessian based approximation of the standard errors with estimated parameters.

90% CI d=0.8

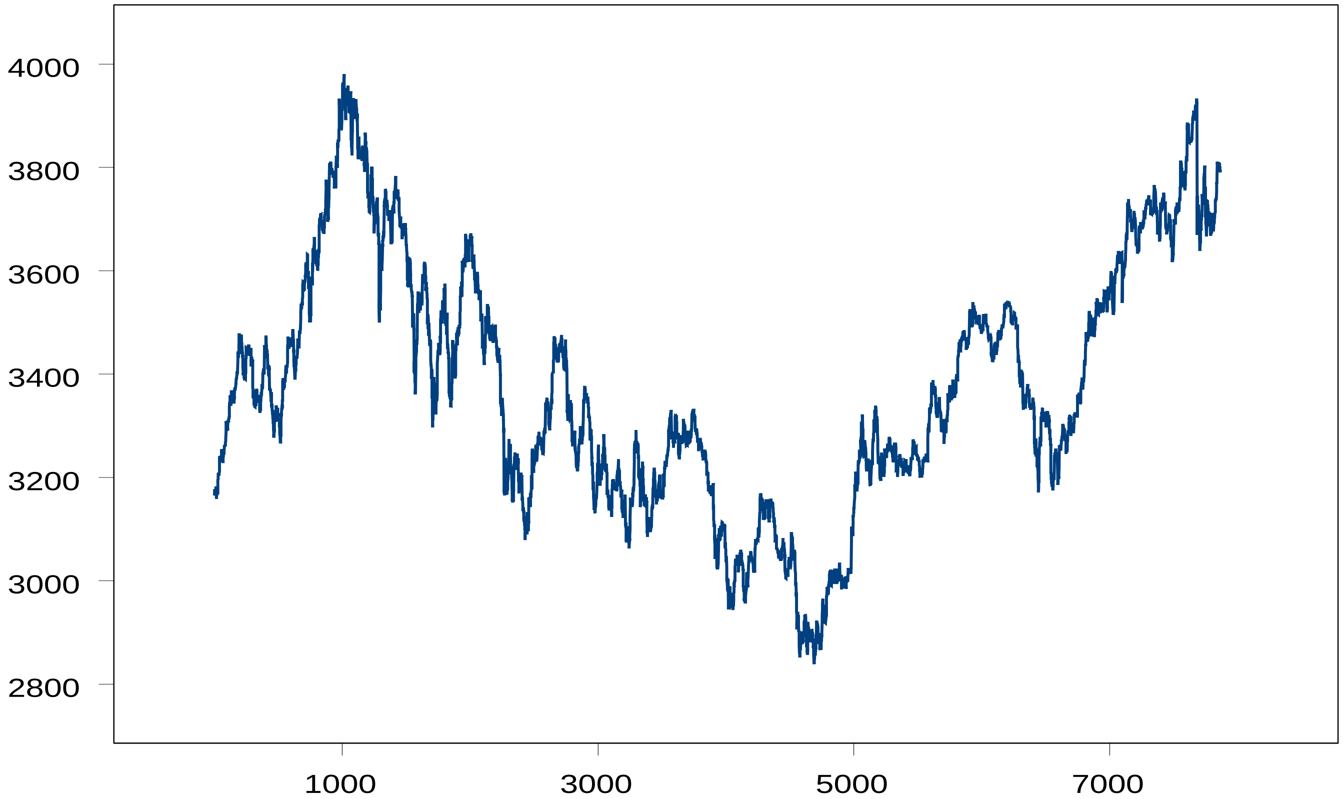
Table 5: 90% Confidence Intervals ($d_0 = 0.8$)

		$m = n^{0.4}$					$m = n^{0.6}$					$m = n^{0.8}$					$m = m_{est}^{opt}$				
σ_w^2		LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE	LPE	GSE	NLPE	ALPE	MGSE
											$n = 1024$										
0.5	Prob.A	0.891	0.859	1.000	1.000	1.000	0.448	0.294	0.900	0.759	0.731	0.000	0.000	0.182	0.547	0.543	0.863	0.768	0.979	0.436	0.444
	Mean.A	0.527	0.411	0.895	0.890	0.676	0.264	0.206	0.435	0.428	0.337	0.132	0.103	0.239	0.212	0.166	0.406	0.336	0.475	0.150	0.117
	Med.A	0.527	0.411	0.809	0.796	0.640	0.264	0.206	0.426	0.418	0.331	0.132	0.103	0.237	0.210	0.165	0.406	0.336	0.467	0.148	0.117
	Prob.H	0.940	0.911	0.996	1.000	1.000	0.511	0.343	0.975	0.998	0.965	0.000	0.000	0.125	0.945	0.884	0.921	0.897	0.994	0.900	0.862
	Mean.H	0.690	0.538	1.522	231.83	1.270	0.294	0.229	0.496	0.670	0.504	0.137	0.107	0.200	0.426	0.330	0.492	0.413	0.579	0.380	0.296
	Med.H	0.690	0.538	1.289	1.503	1.184	0.294	0.229	0.464	0.639	0.499	0.137	0.107	0.198	0.415	0.328	0.492	0.413	0.532	0.370	0.292
0.1	Prob.A	0.787	0.657	1.000	1.000	1.000	0.023	0.006	0.829	0.593	0.569	0.000	0.000	0.024	0.414	0.393	0.841	0.783	1.000	0.276	0.289
	Mean.A	0.527	0.411	0.971	1.012	0.727	0.264	0.206	0.470	0.438	0.345	0.132	0.103	0.430	0.216	0.169	0.609	0.496	0.776	0.152	0.119
	Med.A	0.527	0.411	0.822	0.801	0.652	0.264	0.206	0.456	0.415	0.331	0.132	0.103	0.277	0.210	0.166	0.609	0.496	0.720	0.148	0.117
	Prob.H	0.867	0.773	0.992	1.000	1.000	0.027	0.007	0.851	0.986	0.930	0.000	0.000	0.023	0.949	0.948	0.920	0.877	0.994	0.952	0.929
	Mean.H	0.690	0.538	1.503	1152.43	160.502	0.294	0.229	0.473	0.923	0.701	0.137	0.107	0.302	0.697	0.535	0.842	0.698	1.133	0.644	0.499
	Med.H	0.690	0.538	1.278	1.649	1.354	0.294	0.229	0.446	0.845	0.678	0.137	0.107	0.210	0.648	0.517	0.842	0.698	0.974	0.605	0.484
											$n = 4096$										
0.5	Prob.A	0.923	0.778	1.000	1.000	0.991	0.636	0.499	0.865	0.763	0.809	0.000	0.000	0.062	0.592	0.594	0.823	0.824	0.862	0.436	0.440
	Mean.A	0.406	0.317	0.647	0.644	0.505	0.174	0.136	0.281	0.279	0.219	0.076	0.059	0.133	0.122	0.095	0.237	0.197	0.257	0.075	0.060
	Med.A	0.406	0.317	0.627	0.623	0.496	0.174	0.136	0.279	0.277	0.218	0.076	0.059	0.133	0.121	0.095	0.237	0.197	0.256	0.075	0.060
	Prob.H	0.946	0.867	1.000	1.000	1.000	0.674	0.532	0.893	0.896	0.895	0.000	0.000	0.040	0.841	0.831	0.867	0.858	0.879	0.809	0.807
	Mean.H	0.492	0.383	1.004	1.048	0.778	0.185	0.144	0.305	0.368	0.272	0.077	0.060	0.110	0.204	0.158	0.261	0.218	0.269	0.175	0.137
	Med.H	0.492	0.383	0.874	0.957	0.757	0.185	0.144	0.292	0.351	0.271	0.077	0.060	0.109	0.203	0.158	0.261	0.218	0.263	0.173	0.137
0.1	Prob.A	0.893	0.800	1.000	1.000	0.985	0.043	0.012	0.865	0.695	0.690	0.000	0.000	0.000	0.438	0.446	0.813	0.788	0.895	0.322	0.319
	Mean.A	0.406	0.317	0.655	0.653	0.513	0.174	0.136	0.291	0.281	0.220	0.076	0.059	0.151	0.122	0.095	0.352	0.291	0.401	0.075	0.060
	Med.A	0.406	0.317	0.634	0.630	0.502	0.174	0.136	0.289	0.277	0.218	0.076	0.059	0.151	0.121	0.095	0.352	0.291	0.396	0.075	0.060
	Prob.H	0.922	0.872	0.999	1.000	1.000	0.051	0.015	0.861	0.947	0.884	0.000	0.000	0.000	0.880	0.874	0.909	0.853	0.944	0.865	0.851
	Mean.H	0.492	0.383	1.008	1.103	0.804	0.185	0.144	0.282	0.429	0.332	0.077	0.060	0.112	0.303	0.234	0.412	0.345	0.489	0.278	0.217
	Med.H	0.492	0.383	0.883	0.984	0.784	0.185	0.144	0.276	0.419	0.330	0.077	0.060	0.112	0.299	0.233	0.412	0.345	0.432	0.274	0.215
											$n = 8192$										
0.5	Prob.A	0.835	0.827	0.999	0.999	0.997	0.718	0.591	0.859	0.799	0.813	0.000	0.000	0.042	0.593	0.575	0.846	0.840	0.879	0.448	0.447
	Mean.A	0.352	0.274	0.556	0.555	0.436	0.142	0.110	0.228	0.227	0.178	0.057	0.045	0.099	0.092	0.072	0.182	0.151	0.190	0.056	0.046
	Med.A	0.352	0.274	0.547	0.545	0.432	0.142	0.110	0.227	0.226	0.178	0.057	0.045	0.099	0.092	0.072	0.182	0.151	0.190	0.056	0.046
	Prob.H	0.973	0.890	0.999	0.997	1.000	0.740	0.622	0.876	0.882	0.881	0.000	0.000	0.033	0.809	0.776	0.867	0.862	0.883	0.782	0.792
	Mean.H	0.412	0.321	0.888	0.831	0.622	0.148	0.116	0.246	0.286	0.208	0.058	0.045	0.082	0.144	0.112	0.195	0.162	0.191	0.123	0.097
	Med.H	0.412	0.321	0.722	0.769	0.615	0.148	0.116	0.236	0.270	0.207	0.058	0.045	0.082	0.144	0.112	0.195	0.162	0.189	0.122	0.097
0.1	Prob.A	0.825	0.796	0.998	0.998	0.994	0.067	0.023	0.872	0.725	0.721	0.000	0.000	0.000	0.440	0.452	0.807	0.782	0.877	0.306	0.341
	Mean.A	0.352	0.274	0.559	0.558	0.438	0.142	0.110	0.233	0.228	0.178	0.057	0.045	0.112	0.092	0.072	0.268	0.222	0.294	0.056	0.046
	Med.A	0.352	0.274	0.548	0.546	0.433	0.142	0.110	0.232	0.227	0.177	0.057	0.045	0.112	0.092	0.072	0.268	0.222	0.292	0.056	0.046
	Prob.H	0.960	0.873	0.999	0.999	1.000	0.076	0.026	0.867	0.873	0.873	0.000	0.000	0.000	0.825	0.833	0.860	0.841	0.894	0.829	0.807
	Mean.H	0.412	0.321	0.812	0.816	0.632	0.148	0.116	0.226	0.313	0.242	0.058	0.045	0.083	0.210	0.163	0.299	0.250	0.328	0.191	0.149
	Med.H	0.412	0.321	0.720	0.771	0.624	0.148	0.116	0.223	0.309	0.242	0.058	0.045	0.083	0.208	0.162	0.299	0.250	0.304	0.189	0.149

Prob.A, Mean.A, Med.A denote coverage percentages, mean lengths and median lengths of the nominal 90% confidence intervals with the asymptotic expression for standard errors with estimated parameters. Prob.H, Mean.H, Med.H denote coverage percentages, mean lengths and median lengths of the nominal 90% confidence intervals with the finite sample Hessian based approximation of the standard errors with estimated parameters.

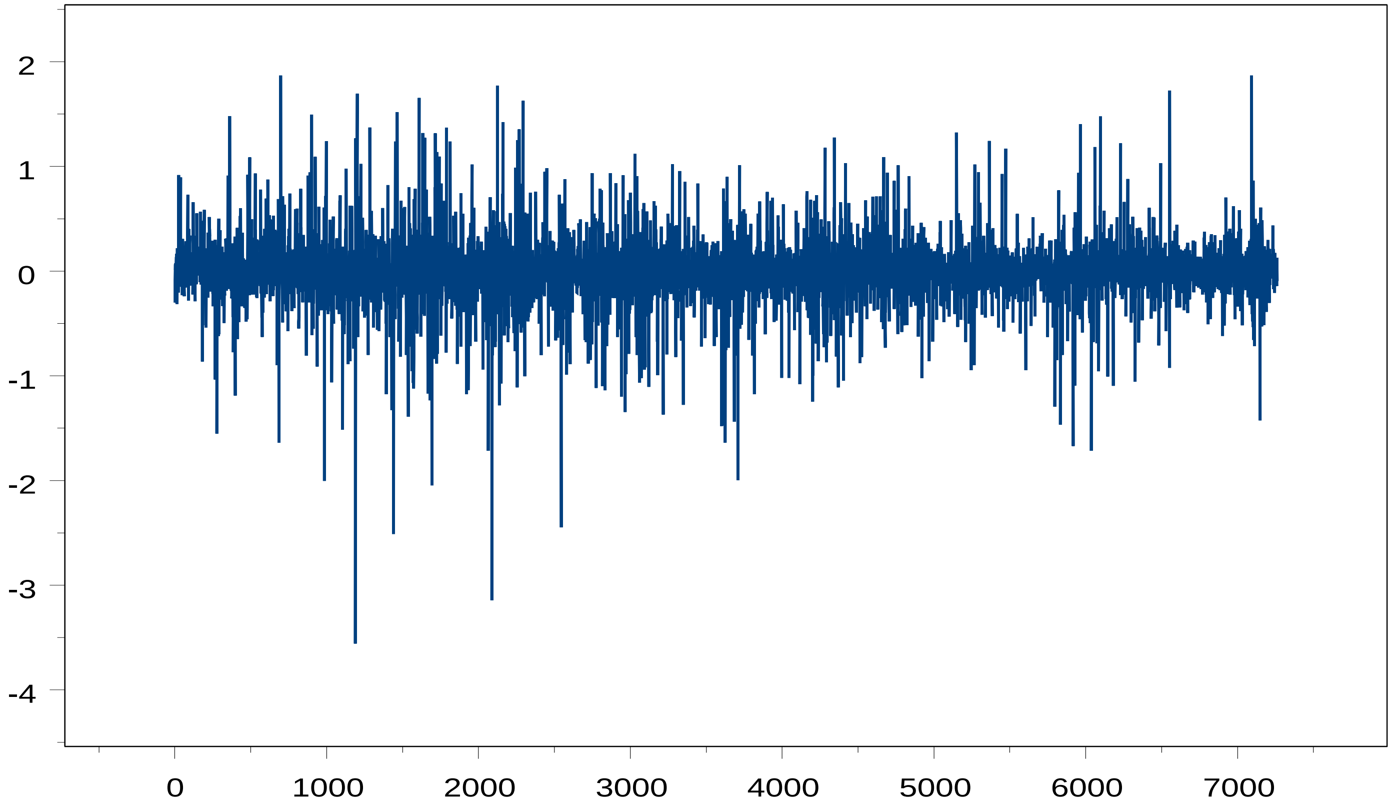
Ibex35

The series of returns covers the period 1-10-93 to 22-3-96 half-hourly.

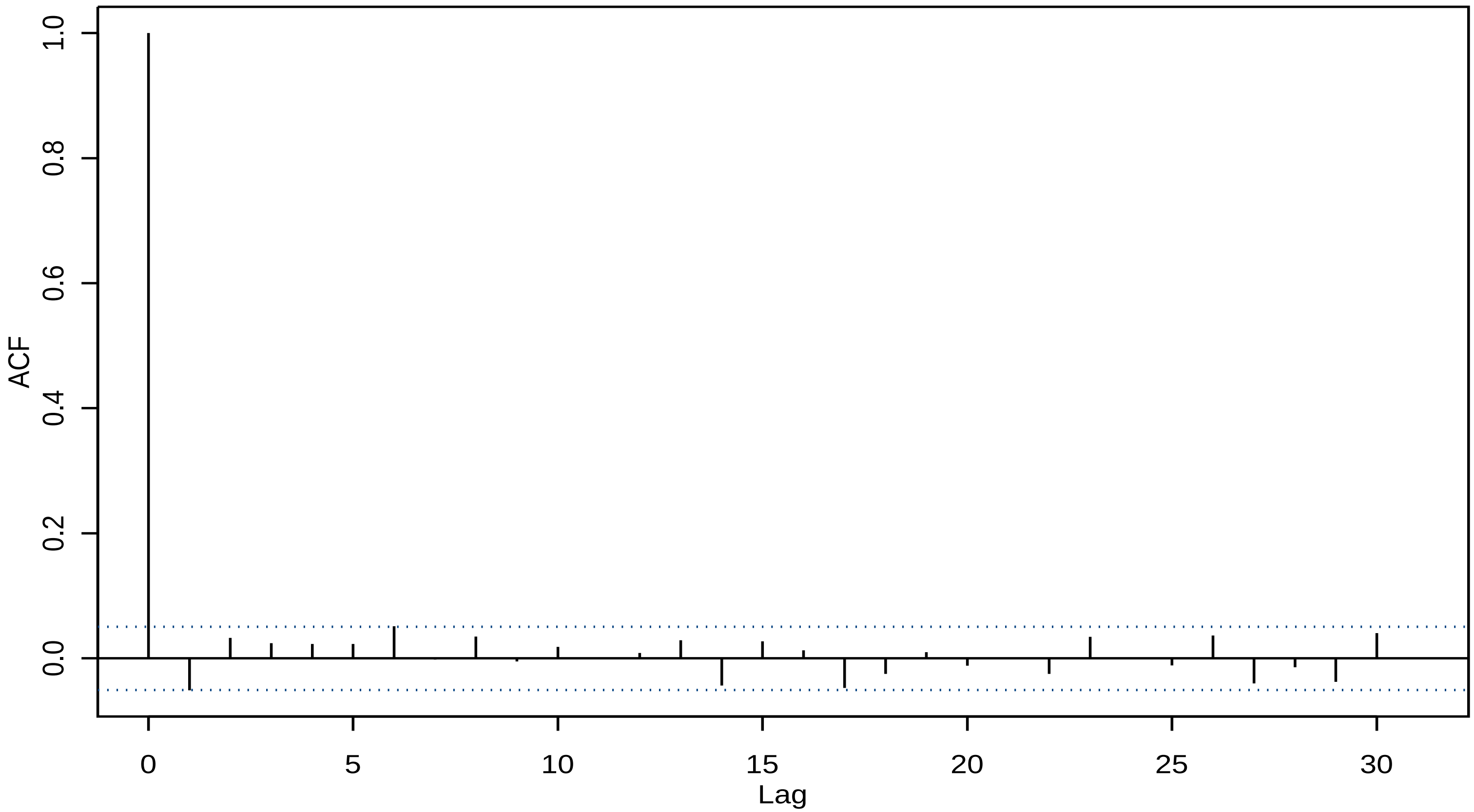


Ibex35

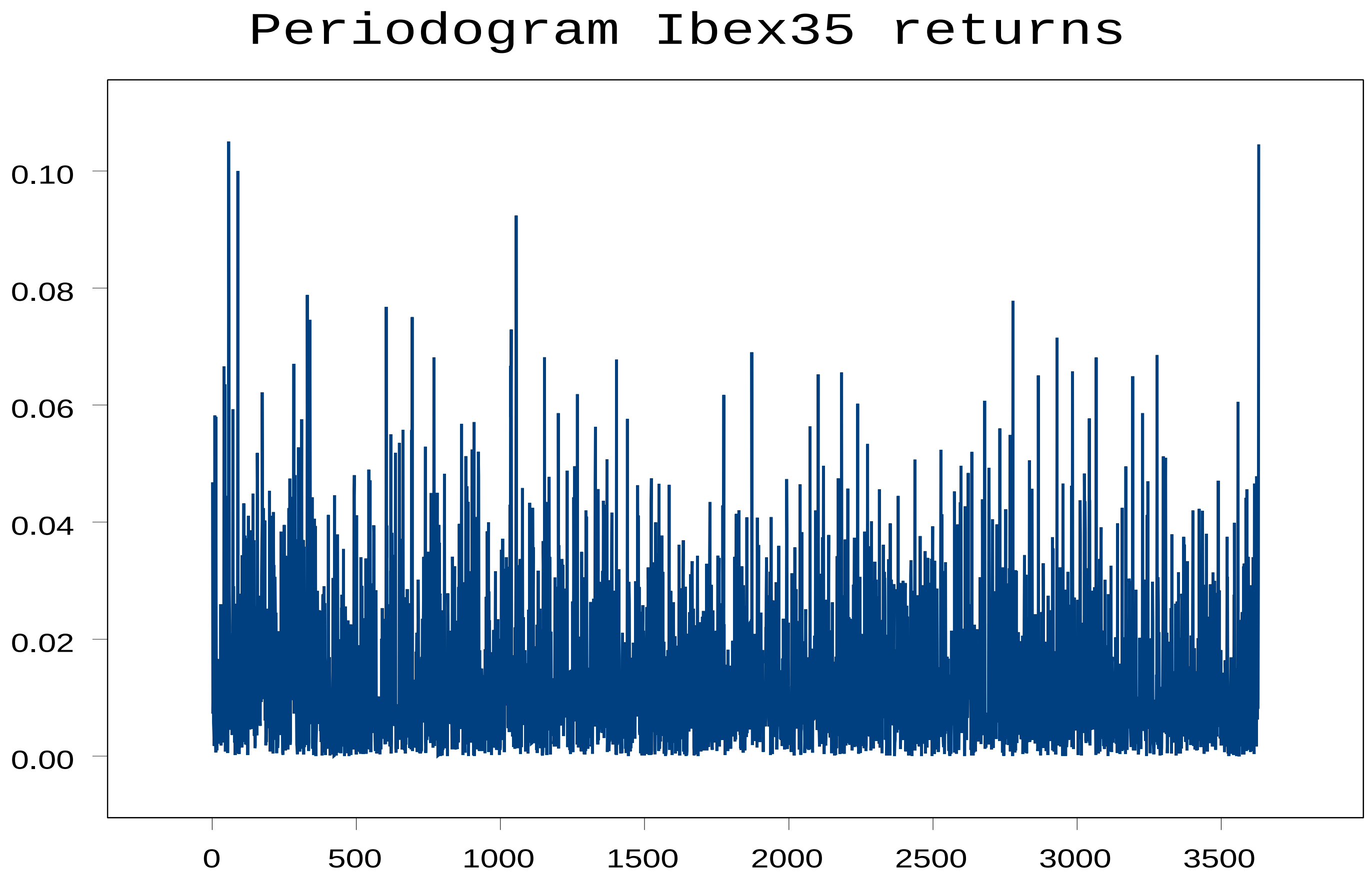
The series of returns covers the period 1-10-93 to 22-3-96 half-hourly.



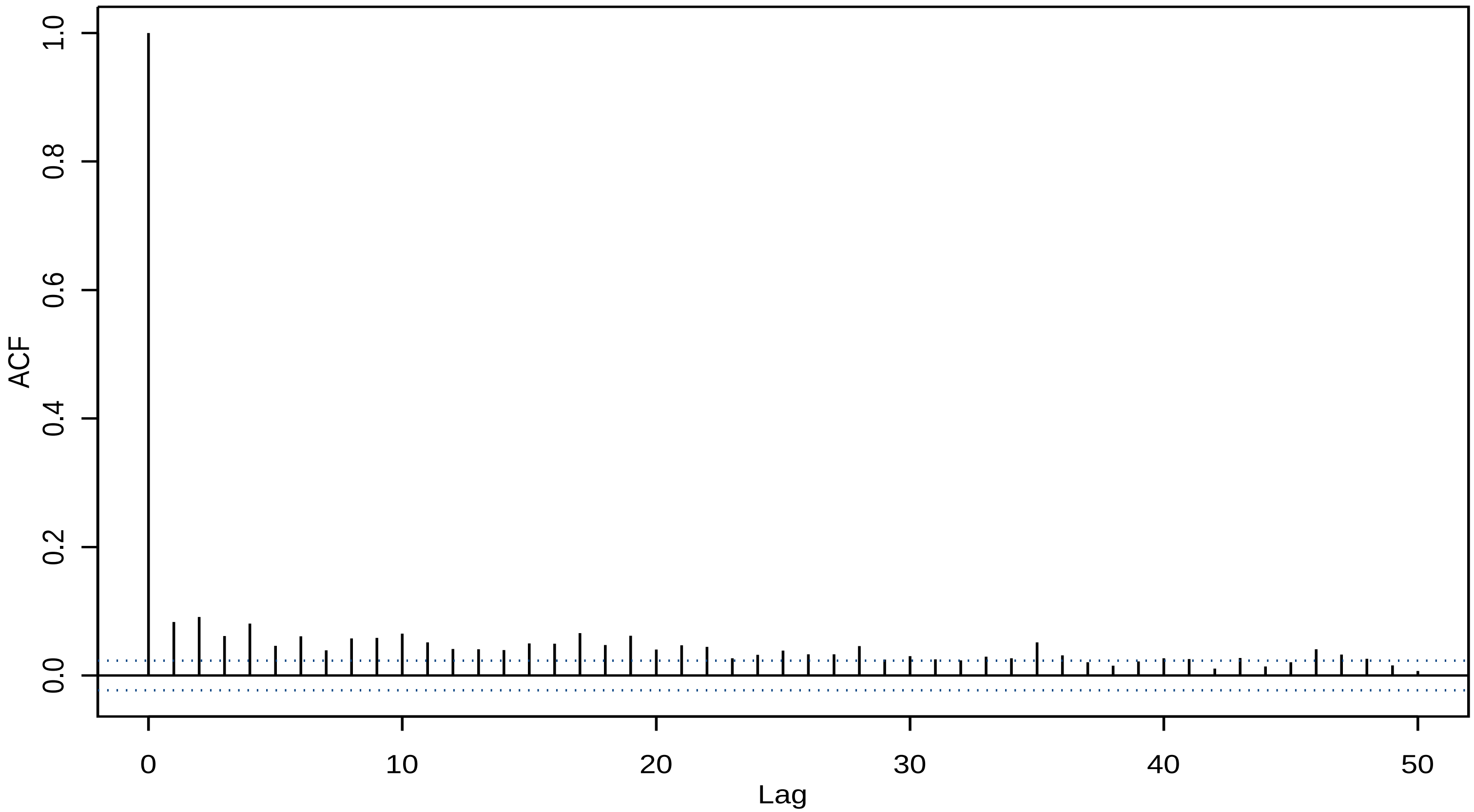
Ibex35: autocorrelations of returns



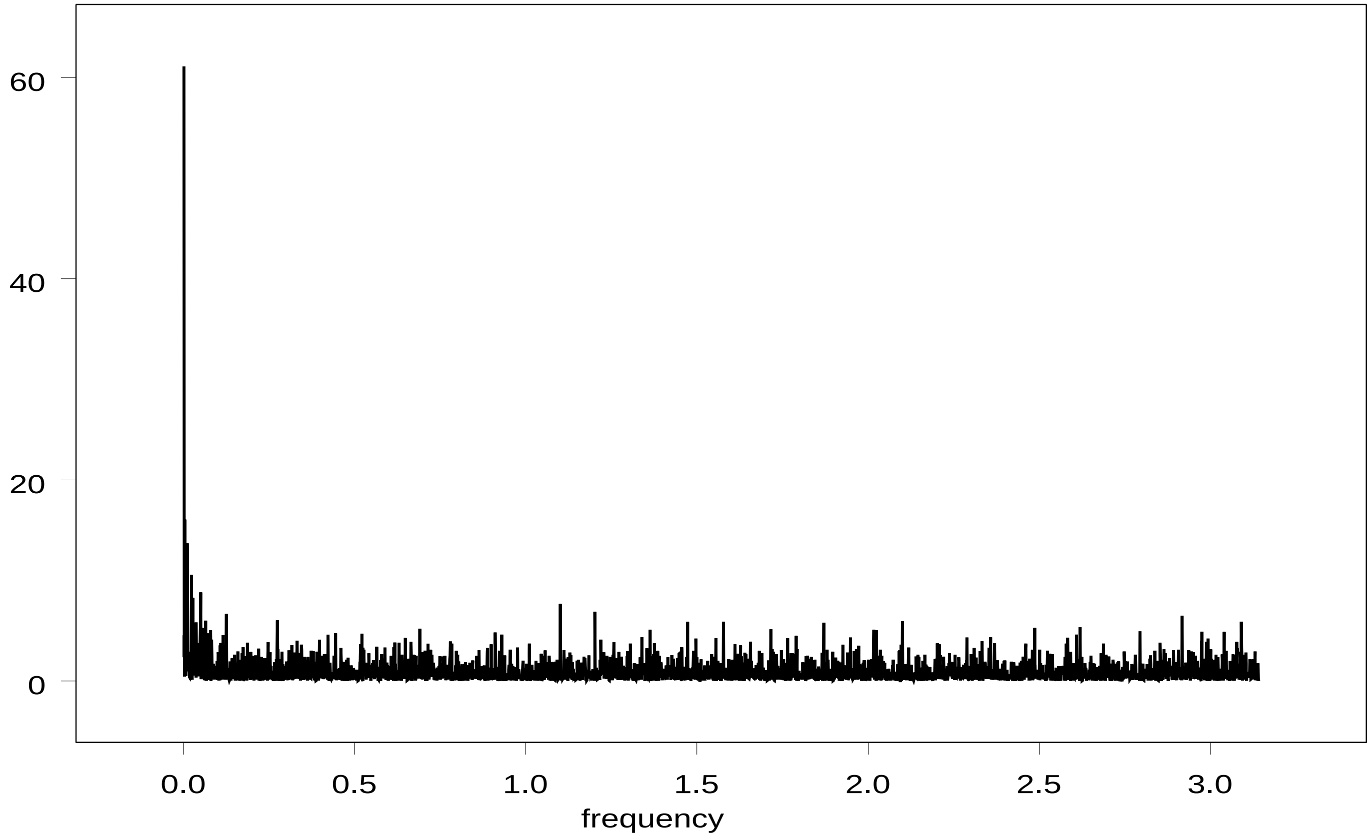
Ibex35: periodogram of returns



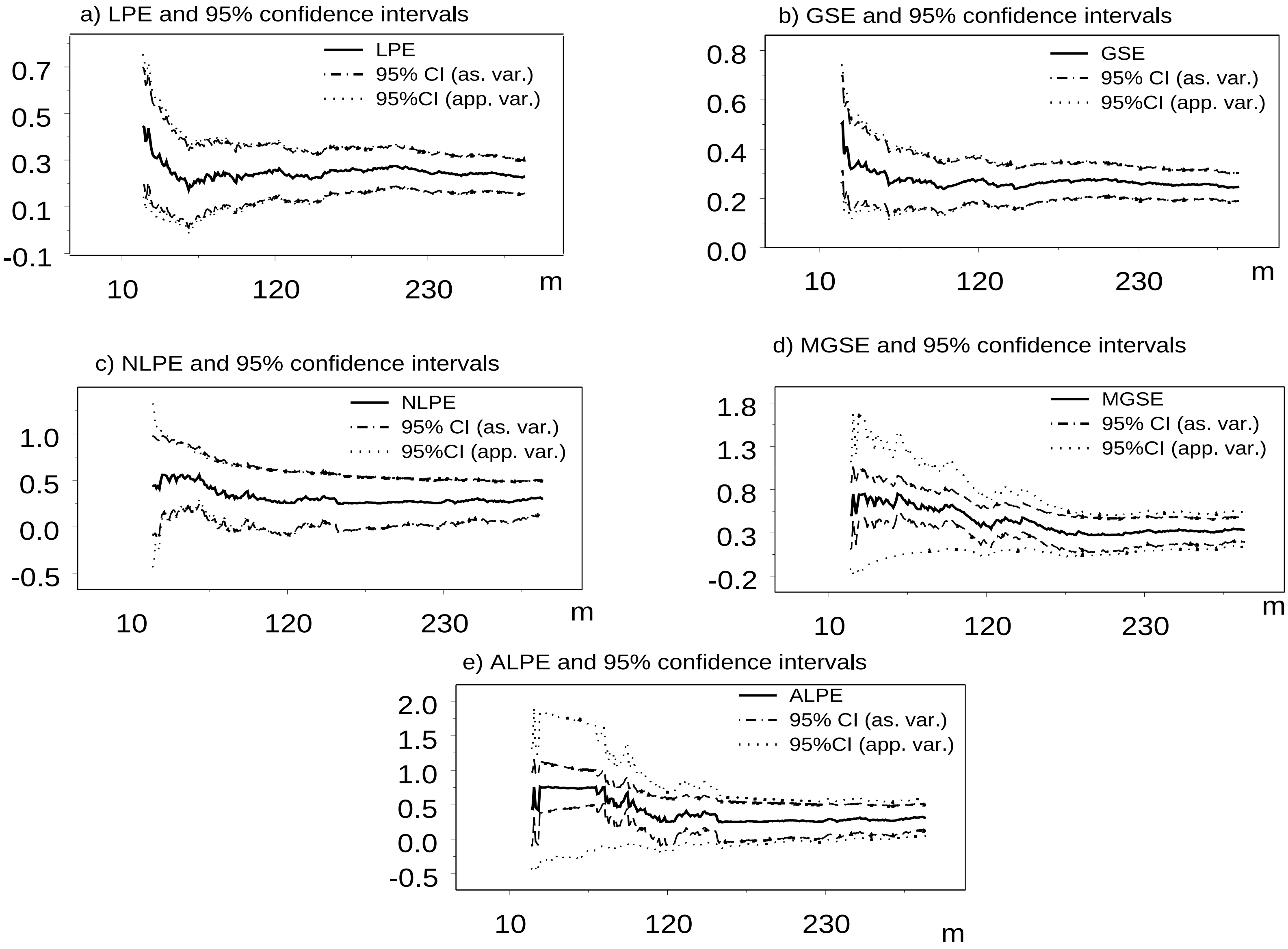
Ibex35: autocorrelations of volatility



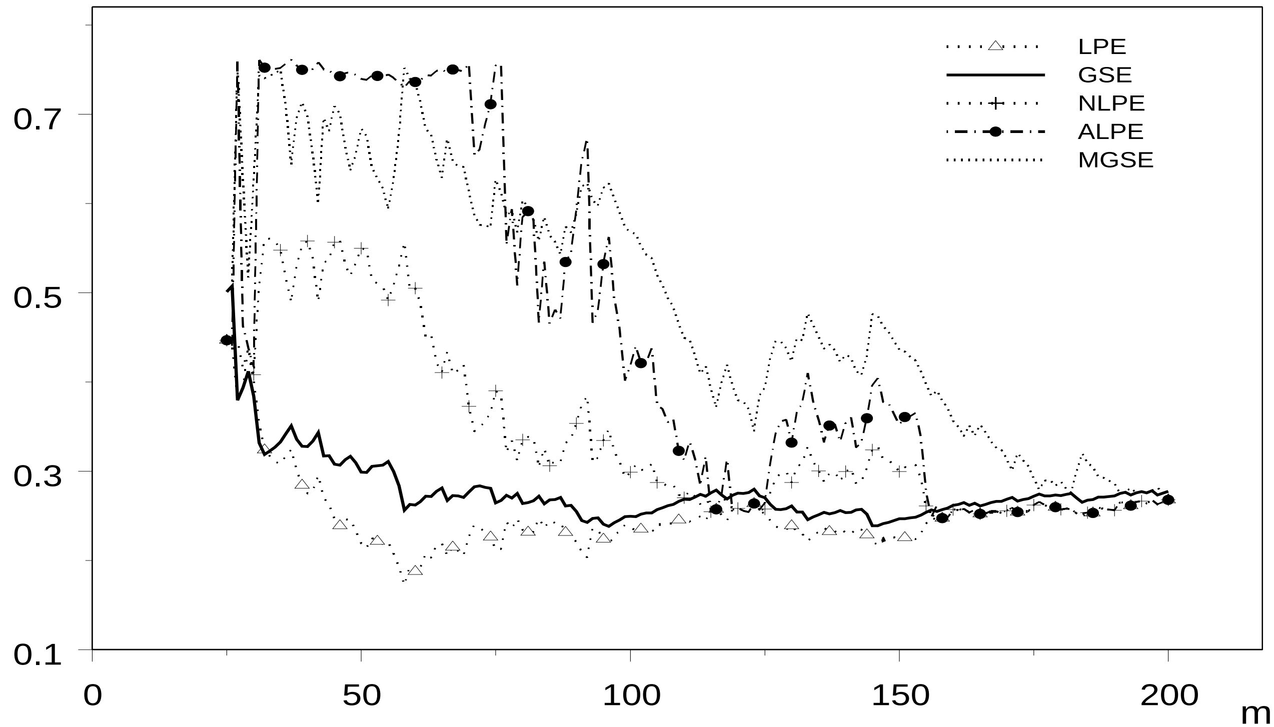
Ibex35: periodogram of volatility



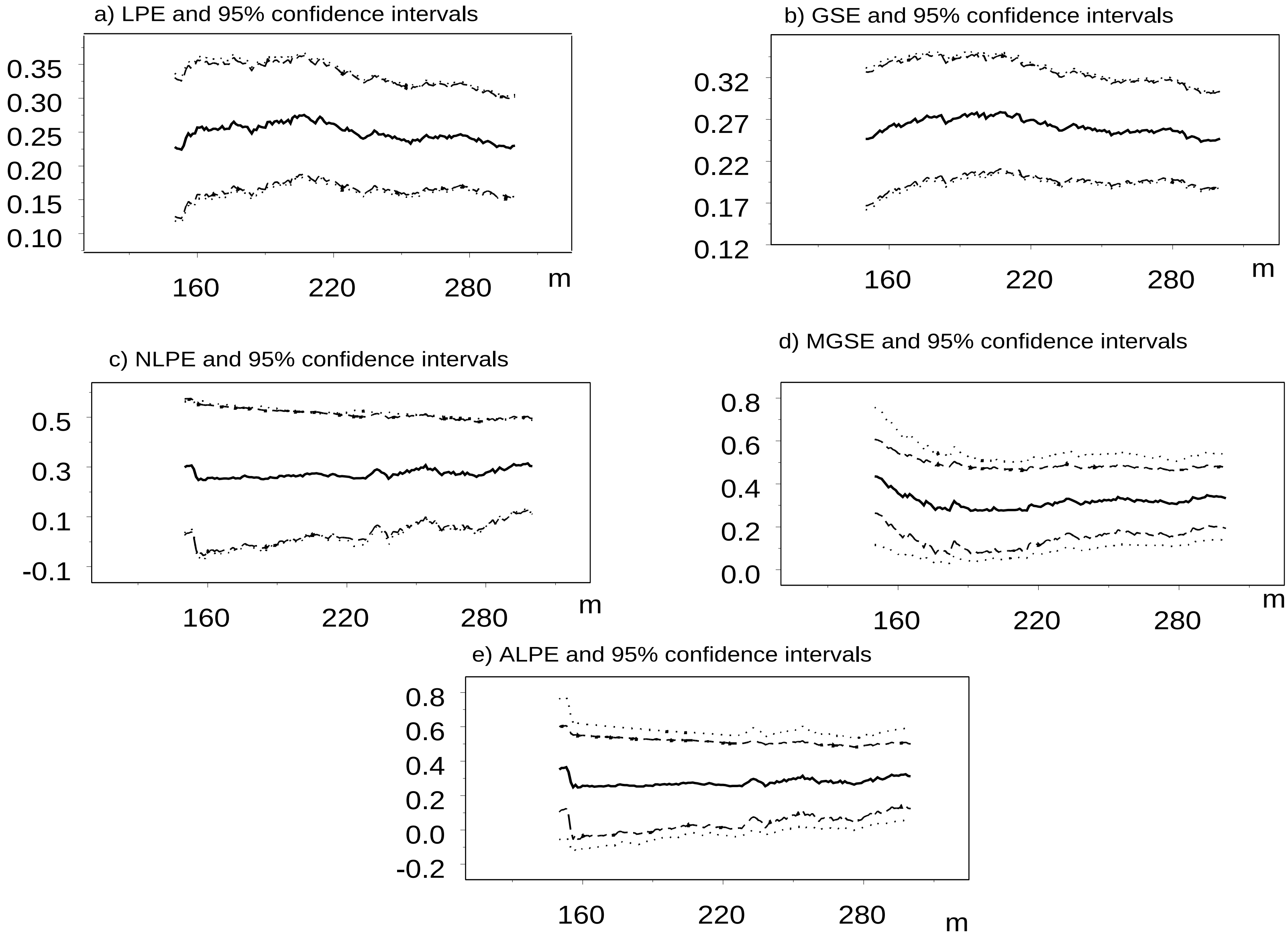
Ibex35: estimates of memory parameter of volatility



Ibex35: estimates of memory parameter of volatility



Ibex35: estimates of memory parameter of volatility



Conclusions

- If the added noise is not considered explicitly in the estimation the LPE and GSE can render meaningless estimates.
- More reliable are the NLPE (with low n and m) and the ALPE and MGSE.

Conclusions and extensions

- If the added noise is not considered explicitly in the estimation the LPE and GSE can render meaningless estimates.
- More reliable are the NLPE (with low n and m) and the ALPE and MGSE.
- To be proved: properties of ALPE when u_t, y_t are not Gaussian and/or $d \geq 0.5$
- Proposal of feasible versions of m^{opt}.
- Extensions for correlated signal and noise.